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On the Variety of Smooth Rational Space Curves with Given Degree and Normal Bundle

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1. Introduction

In this paper we shall show that the family of smooth rational curves in $\mathbb{P}_3 = \mathbb{P}_3(\mathbb{C})$ with given degree and normal bundle is irreducible and unirational, and we compute its dimension.

For convenience we shall work with parametrised curves. More precisely, we consider the space $\mathbb{P}_{4n+3}$ of 4-tuples of homogeneous polynomials in two variables, of degree n, modulo scalars. Let $S_n \subset \mathbb{P}_{4n+3}$ be the subset consisting of those 4-tuples $f=(f_0, \ldots, f_3)$ for which the map $f\colon \mathbb{P}_1 \to \mathbb{P}_3$ is an isomorphism onto its image. The normal bundle of f (or of $f(\mathbb{P}_1)$) is by definition the 2-vector bundle $N_f = f^*(T_{\mathbb{P}_3})/T_{\mathbb{P}_1}$. In [3] we proved that for each f there is an integer a with $0 \leqq a \leqq n-4$, such that the bundle N_f splits as $N_f \cong \mathcal{O}_{\mathbb{P}_1}(2n-1-a) \oplus \mathcal{O}_{\mathbb{P}_1}(2n-1+a)$, and that every a in this range occurs for some f. Let $S_{n,a} \subset S_n$ be the set consisting of those f for which

$$N_f \cong \mathcal{O}_{\mathbb{P}_1}(2n-1-a) \oplus \mathcal{O}_{\mathbb{P}_1}(2n-1+a).$$

Main Theorem. *For all a with $0 \leqq a \leqq n-4$, the set $S_{n,a}$ is a non-empty, locally closed, irreducible and rational subvariety of S_n. The variety $S_{n,0}$ is dense in S_n, whereas* $\dim S_{n,a} = 4n-2a+4$ *if $a \geqq 1$. The closure of $S_{n,a}$ is $S_{n,a} \cup S_{n,a+1} \cup \ldots$.*

If one wishes to consider the variety $\tilde{S}_n$ of (unparametrised) smooth rational curves of degree n in $\mathbb{P}_3$, one must take the quotient of S_n by $\mathrm{Aut}(\mathbb{P}_1) = \mathbb{P}\mathbf{GL}(2, \mathbb{C})$. This group acts without fixed points on S_n (and on each $S_{n,a}$), preserving an open affine covering, so that a quotient with good properties exists.

We note that G. Sacchiero ([5]) has proven part of the preceding result, and also that the $S_{n,a}$ are locally Cohen-Macaulay.

Our proof depends on knowing (or in any case bounding) the dimension of the space of parametrised rational curves of given degree in $\mathbb{P}_3$ with at least a given number of cusps (counted with multiplicities, as explained below). Such a bound can be deduced from Martens' Theorem bounding the dimensions of

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the varieties of special divisors, generalised to singular curves. This was done in the original version of this paper. In Sect. 2 below we deduce, following [2], the exact dimension from a result on the transversality of Schubert cycles associated to osculating flags at distinct points on a rational normal curve.

We are grateful to Hulek for showing us his algebraic proof for a special case of Proposition 3.2; it has led to a simplification of our fundamental construction (Sect. 5).

Section 3 is elementary, and devoted to the proof of certain facts from the "differential geometry" of curves in $\mathbb{P}_3$, which we need later. In Sect. 4 we exploit the existence and properties, given by Brieskorn, of the versal family of Hirzebruch surfaces, to bound from below the dimension of components of $S_{n,a}$. In Sect. 6 we describe a "generic" part $S'_{n,a}$ of $S_{n,a}$, which is irreducible, rational and of the right dimension. In Sect. 7 we then complete the proof of the main theorem by showing that $S_{n,a}-S'_{n,a}$ is too small to harbour any components of $S_{n,a}$.

Among the open questions that remain, the most interesting is perhaps whether $\tilde{S}_{n,a}$ (the quotient of $S_{n,a}$ by $\mathrm{Aut}(\mathbb{P}_1)$) is again rational. It follows easily from our result that $\tilde{S}_{n,a}$ is *uni*rational, and it does not seem very difficult to show that $\tilde{S}_{n,a}$ is rational over the variety of all rational space curves of a certain degree, but we don't know whether this last variety is rational.

No doubt it is possible to extend the results of this paper to rational curves in any $\mathbb{P}_N$; but it is to be feared that this generalisation is rather straightforward. In a different direction, it seems reasonable to hope that our technique of studying curves on tangent surfaces may provides a useful source of examples for (smooth) curves in $\mathbb{P}_3$. And it also might be interesting to identify the sets $S_{n,a}$ in some other way, say, using invariant theory or simple projective geometry.

We now introduce some basic notation and terminology. By a *curve* we mean a reduced, irreducible 1-dimensional projective scheme, algebraic over $\mathbb{C}$. Points will always be closed points. A *parametrised rational curve* or simply a *parametrised curve* in $\mathbb{P}_3$ is a non-constant map $f\colon \mathbb{P}_1 \to \mathbb{P}_3$, which we usually identify with a 4-tuple of homogeneous polynomials (modulo scalars) in two variables t_0, t_1, having no common zeros. If $\lambda \in \mathbb{P}_1$, then in suitable affine coordinates near λ and $f(\lambda)$ we can write f in the form $f(t)=(t^{1+k_0}+\text{higher order terms}, t^{2+k_0+k_1}+\dots, t^{3+k_0+k_1+k_2}+\dots)$. The integers $k_0=k_0(f,\lambda)$, $k_1=k_1(f,\lambda)$ and $k_2=k_2(f,\lambda)$ are independent of the coordinates chosen. We set $k_i(f)=\sum_\lambda k_i(f,\lambda)$ for $i=1,2,3$, and define *cusp*(f) to be the divisor $\sum_\lambda k_0(f,\lambda)\lambda$, calling $k_0(f)$ the *number of cusps* of f.

From any parametrised rational curve $f\colon \mathbb{P}_1 \to \mathbb{P}_3$ we can construct two other parametrised curves $f^{(1)}\colon \mathbb{P}_1 \to G$, where G is the Grassmannian of lines in $\mathbb{P}_3$, and $f^{(2)}=f^{\vee}\colon \mathbb{P}_1 \to \mathbb{P}_3^{\vee}$, in the following way. We take $f^{(1)}(\lambda)$ and $f^{(2)}(\lambda)$ to be the tangent line and the osculating plane of $f(\mathbb{P}_1)$ at $f(\lambda)$ if $f(\lambda)$ is a smooth point of $f(\mathbb{P}_1)$, and we define $f^{(1)}(\lambda)$ and $f^{(2)}(\lambda)$ for singular $f(\lambda)$ by requiring $f^{(1)}$ and $f^{(2)}$ to be continuous. Of course $f^{(2)}$ is non-constant only if $f(\mathbb{P}_1)$ is non-planar, and we shall consider $f^{(2)}$ only in such cases. We regard G

as embedded in $\mathbb{P}_5$ by the Plücker embedding, and define

$$d = d^0(f) = \deg(f) \quad = \deg f^*(\mathcal{O}_{\mathbb{P}_3}(1)),$$
$$d^{(1)} = d^1(f) = \deg(f^{(1)}) = \deg(f^{(1)})^*(\mathcal{O}_G(1)),$$
$$d^{(2)} = d^2(f) = \deg(f^{(2)}) = \deg(f^{(2)})^*(\mathcal{O}^{\vee}_{\mathbb{P}_3^{\vee}}(1)).$$

Thus $d^1(f)$ is the number of tangent lines of $f(\mathbb{P}_1)$, meeting a general line, multiplied by the degree of the map $f\colon \mathbb{P}_1 \to f(\mathbb{P}_1)$, etc.

Remark. When we speak of parametrised rational curves in general, we shall denote their degree by d; whereas we shall call this degree n when we directly study the sets $S_{n,a}$.

It is known that, if $f(\mathbb{P}_1)$ is not planar, then $f^{\vee\vee} = f$. Furthermore, $f^{(1)}$ and $f^{\vee(1)}$ correspond under the natural duality between G and $G^{\vee}$, where $G^{\vee}$ is the Grassmann variety of lines in $\mathbb{P}_3^{\vee}$. (See [4] for details and more abstract definitions).

2. Counting Curves with Cusps

For each triple of integers: $s \geqq 0$, $r \geqq 2$ and $d \geqq r$, we compute the dimension of the space of non-degenerate parametrised curves $f\colon \mathbb{P}_1 \to \mathbb{P}_r$ of degree d having $k_0(f) \geqq s$, that is, curves with at least s cusps.

Each parametrised curve $f\colon \mathbb{P}_1 \to \mathbb{P}_r$ of degree $d \geqq r$ can be factorised as $f = g \circ n$, where $n\colon \mathbb{P}_1 \to \mathbb{P}_d$ is an isomorphism onto a rational normal curve, spanning $\mathbb{P}_d$, and $g\colon n(\mathbb{P}_1) \to \mathbb{P}_r$ the projection from a linear subspace $\mathbb{P}_{d-r-1} \subset \mathbb{P}_d$, which does not meet $n(\mathbb{P}_1)$. This factorisation is far from unique. We shall refer to it by simply saying that f is obtained as "projection of a rational normal curve".

It is easy to see, by considering our curves as projections of a rational normal curve, that for a given $\lambda \in \mathbb{P}_1$, the set of curves f with $k_0(f, \lambda) \geqq 1$ has codimension r in the space of all non-degenerate rational curves of degree d, thus the set of curves f with $k_0(f) \geqq 1$ has codimension $r-1$. For $s > 1$ however, there is a problem of the independence of such conditions. This problem is solved by the following proposition from [2]. (In a dual form, the result was first discovered by A. Iarrobino in 1973, but not published by him.)

Proposition 2.1. *Let $C \subset \mathbb{P}_d$ be a rational normal curve (of degree d), and let $p_1, \ldots, p_s \in C$. Let k be an integer with $0 \leqq k \leqq d-1$. For $i = 1, \ldots, s$, let σ_i be a Schubert cycle in the Grassmannian $G(k,d)$ of k-planes in $\mathbb{P}_d$, defined with respect to the flag of osculating planes to C at p_i. If $p_1, \ldots, p_s$ are distinct points, then*

$$\operatorname{cod} \bigcap_{i=1}^{s} \sigma_i = \bigcap_{i=1}^{s} \operatorname{cod} \sigma_i,$$

unless the product of the fundamental classes of the σ_i vanishes in $H^*(G(k,d), \mathbb{Z})$, in which case $\bigcap_{i=1}^{s} \sigma_i = \emptyset$.

If we regard a non-degenerate parametrised curve $f\colon \mathbb{P}_1 \to \mathbb{P}_r$ of degree d as the projection of a rational normal curve C in $\mathbb{P}_d$, then the condition $k_i(f, \lambda)$ for a point $\lambda \in \mathbb{P}_1$ can be expressed as a Schubert condition with respect to the osculating flag at $n(\lambda) \in C$. One may thus deduce, as in [2].

Theorem 2.2. *Let $\lambda_1, \ldots, \lambda_s \in \mathbb{P}_1$, and let $\alpha_1, \ldots, \alpha_s$ be non-negative integers. The space of non-degenerate parametrised curves $f\colon \mathbb{P}_1 \to \mathbb{P}_r$ of degree $d>r$ with $\mathrm{cusp}(f) \geqq \sum_{i=1}^{s} \alpha_i f_i$ has dimension $(r+1)(d+1) - r\sum \alpha_i$ if $\sum \alpha_i \leqq (d-r)(r+1)/r$ and is empty otherwise. Thus the space of non-degenerate parametrised curves $f\colon \mathbb{P}_1 \to \mathbb{P}_r$ of degree d with $k_0(f) \geqq s$ has dimension $(r+1)(d+1) - rs + s$ if $s \leqq (d-r)(r+1)/r$ and is empty otherwise.*

3. Some Elementary Complex Differential Geometry

The remarks in this section are not very deep, but they play nevertheless an essential role in our proof.

First a word about the *tangent surface* T of a (non-planar) space curve C. This surface consists of those points of $\mathbb{P}_3$, which are contained in at least one tangent line to C (in singular points the tangent line has to be taken as a limit, or by extension of the Gauss map). If $\tilde{C}$ is the normalisation of C, then there is an obvious map $g\colon \tilde{C} \to G$ (the Grassmannian of lines in $\mathbb{P}_3$), namely the Gauss map, and the (smooth) normalisation $\tilde{T}$ of T is the pull-back under g of the projective bundle associated to the universal subbundle. The normalisation map $\tilde{T} \to T$ is finite (compare [3], Sect. 4).

Next we present a couple of simple criteria for a curve to lie on a certain surface.

Proposition 3.1. *Let C be a space curve, not contained in a plane, and suppose that there exists a map $f\colon C \to \mathbb{P}_3^\vee$, with the following properties:*

(i) *all planes $f(x)$, $x \in C$, pass through one and the same point $p \in \mathbb{P}_3$;*

(ii) *the tangent line $t_x \subset f(x)$ for all $x \in C$.*

Then C is contained in the cone with vertex p, which has the planes $f(x)$, $x \in C$, as tangent planes along its rulings.

Proof. The map f can't be constant, otherwise C would be a plane curve. So there is indeed an irreducible cone, say K, as described in the proposition, and all tangent lines t_x, $x \in C$, are tangent to K (some may of course be contained in K). They can't all pass through p, so if we project (all but a finite number of) them from p onto a general plane H, we find that the tangent lines to the intersection $K \cap H$ are the projections from the tangent lines to C. This means that the projection of C from p is equal to this intersection $K \cap H$, and C is contained in K.

Proposition 3.2. *Let C and D be two smooth curves, which are mapped by $f\colon C \to \mathbb{P}_3$, $g\colon D \to \mathbb{P}_3$ into $\mathbb{P}_3$, such that both $f(C)$ and $g(D)$ span $\mathbb{P}_3$. If there exists a map $m\colon D \to C$, such that for all $x \in D$ the tangent line to $g(D)$ at $g(x)$ is* *contained in the osculating plane to $f(C)$ at $f(m(x))$, then, for all $x \in D$, the tangent line to $f(C)$ at $f(m(x))$ contains $g(x)$, and so $g(D)$ is contained in the tangent surface of C.*

Proof. It is sufficient to show that the statement is true for some open set on D.

We take a point $p \in D$, such that m and g have maximal rank in p, and f in $m(p)$. Then we can use the same local coordinate t on $f(C)$ about $f(m(p))$ and on $g(D)$ about $g(p)$, such that the tangent line to $g(D)$ at $t \in g(D)$ is contained in the osculating plane to $f(C)$ at $t \in f(C)$. Let $f(D)$ be given locally by $x = f(t)$, $y = g(t)$, $z = h(t)$, and $f(C)$ locally by $x = a(t)$, $y = b(t)$, $z = c(t)$ (with respect to suitable affine coordinates x, y, z). We may assume (if necessary, after moving p a little bit) that the osculating plane to $f(C)$ is given by

$$\begin{vmatrix} x-a & y-b & z-c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0.$$

By assumption we have

$$\begin{vmatrix} f-\mu f'-a & g-\mu g'-b & h-\mu h'-c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \equiv 0 \quad \text{(in } t \text{ and } \mu).$$

Putting

$$\begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix} = T_1, \quad -\begin{vmatrix} a' & c' \\ a'' & c'' \end{vmatrix} = T_2, \quad \text{and} \quad \begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix} = T_3,$$

we have

$$T_1 f' + T_2 g' + T_3 h' \equiv 0 \text{ (in } t)$$

and also

$$T_1(f-a) + T_2(g-b) + T_3(h-c) \equiv 0 \text{ (in } t).$$

Differentiation of the second equation and subtraction gives

$$T_1'(f-a) + T_2'(g-b) + T_3'(h-c) \equiv 0 \text{ (in } t).$$

This means that $t \in f(D)$ is always contained in the plane

$$T_1'(x-a) + T_2'(y-b) + T_3'(z-c) = 0.$$

But this plane meets the osculating plane at $t \in f(C)$ in the tangent line to $f(C)$ at t (for both $a(t)$ and $a(t) + a'(t)$ are contained in it). Since the two planes are different for almost all t we have proved the proposition. (If the two planes were everywhere the same, then the rank of

$$\begin{vmatrix} T_1 & T_2 & T_3 \\ T_1' & T_2' & T_3' \end{vmatrix}$$

would be 0 everywhere. This implies that the ratios T_i/T_j are constant, which is impossible).

Remark 3.3. From the remarks above about the tangent surface and Proposition 3.2 it follows that g always can be lifted to $\tilde{T}_{f(C)}$, even if $g(D)$ is contained in the singular locus of this surface.

Finally we shall use the classical observation that the tangent planes are constant along rulings of the tangent surface, being the osculating plane in the corresponding point of the curve:

Proposition 3.4. *Let C be a non-planar space curve, T its tangent surface and n: $\tilde{T} \to T$ the normalisation. If D is a curve on $\tilde{T}$, then for every point $x \in D$ the tangent to $n(D)$ at $n(x)$ is contained in the osculating plane of C at y, where y is the point on C, determined by the fibre through x.*

4. A Bound on the Dimensions of the Components of $S_{n,a}$

Basic to our proof of the irreducibility of the $S_{n,a}$'s is the following intermediate result.

Proposition 4.1. *For every $a \geqq 0$ the set $S_{n,a} \cup S_{n,a+1} \cup \ldots$ is closed in S_n and each of its components has dimension $\geqq 4n-2a+4$.*

Remark. This remains true if we replace smooth parametrised curves by immersed curves.

Corollary 4.2. *If the components of $S_{n,a+1}$ have dimension $<4n-2a+4$, then $S_{n,a+1}$ is contained in the closure of $S_{n,a}$.*

Proof of Proposition 4.1. Let h: $S_n \times \mathbb{P}_1 \to \mathbb{P}_3$ be defined by $h(f,t)=f(t)$. The tangent bundle along the fibres $f \times \mathbb{P}_1$ is a subbundle of $h^*(T_{\mathbb{P}_3})$. Let N be the quotient bundle. The restriction of N to any $f \times \mathbb{P}_1$ is the normal bundle N_f. Thus the N_f fit together to form an (algebraic) family $\mathcal{N}$ of 2-vector bundles on S_n. Taking the associated $\mathbb{P}_1$-bundle we obtain an algebraic family of Hirzebruch surfaces over S_n, the surface over f being $\mathbb{P}(N_f)$. General theorems yield that $S_{n,a}$ is (Zariski-)locally closed in S_n. By [1], Satz 6.2 for each $f \in S_n$ there is a neighbourhood U_f and an analytic map φ_f: $U_f \to V$, where V is the base space of the versal family of Hirzebruch surfaces, such that $\mathbb{P}(\mathcal{N})|U_f$ is the pull-back of the universal family via φ. The space V is smooth, and if we write V_a for the stratum over which the fibre of the universal family is $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(2n-1-a) \oplus \mathcal{O}_{\mathbb{P}_1}(2n-1+a))$, then the closure of V_a is $V_a \cup V_{a+1} \cup \ldots$, and V_a is irreducible of codimension $2a-1$. Since $S_{n,a} \cap U_f = \varphi_f^{-1}(V_a)$, the result follows.

5. A Basic Construction

We recall that S_n is the space of smooth parametrised curves of degree n in $\mathbb{P}_3$, regarded as an open subset of the projective space $\mathbb{P}_{4n+3}$, whose points are 4-tuples of polynomials of degree n in (t_0, t_1) modulo scalars. We shall denote by

R_m the space of all maps (not necessarily smooth!) of degree m from $\mathbb{P}_1$ into $\mathbb{P}_3^\vee$, regarded again as an open subset of $\mathbb{P}_{4n+3}$.

Now let $a \geqq 1$. If $f=(f_0, \ldots, f_3) \in S_{n,a}$, then we consider the 2-subbundle V_f of $f^*(T_{\mathbb{P}_3})$, which is the pre-image of the unique subbundle of N_f isomorphic to $\mathcal{O}_{\mathbb{P}_1}(2n+a-1)$. From f and V_f we can construct a map $G(f)$: $\mathbb{P}_1 \to \mathbb{P}_3^\vee$, by taking for $G(f)(t)$ the plane in $\mathbb{P}_3$, which is determined by $V_{f,t} \subset T_{\mathbb{P}_3, f(t)}$. It follows from Nakano's theorem (compare [3], § III) that $G(f) \in R_{n-a-1}$. Doing this for all $f \in S_{n,a}$, we obtain a map G: $S_{n,a} \to R_{n-a-1}$.

Theorem 5.1. *The map G is regular, and its fibres are open subsets of linear subspaces in $\mathbb{P}_{4n+3}$.*

The proof of this theorem is based on

Lemma 5.2. *For $f=(f_0, \ldots, f_4) \in S_n$, let j_f: $\mathcal{O}_{\mathbb{P}_1}^2(1) \to \mathcal{O}_{\mathbb{P}_1}^4(n)$ be given by the Jacobian matrix with entries $\frac{\partial f_i}{\partial t_j}$. For any integer $a \geqq 1$ the map $f \in S_{n,a} \cup S_{n,a+1} \cup \ldots$ if and only if there is a map g: $\mathcal{O}_{\mathbb{P}_1}^4 \to \mathcal{O}_{\mathbb{P}_1}(2n-a-1)$ with $g(n) \circ j_f = 0$, where $g(n) = g \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}_1}(n)}$.*

Proof. If we consider the usual diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}_1} & \xrightarrow{n} & \mathcal{O}_{\mathbb{P}_1} & & & & \\
 & & \downarrow{\scriptstyle (t_0, t_1)} & & \downarrow{\scriptstyle (f_0, \ldots, f_4)} & & & & \\
 & & \mathcal{O}_{\mathbb{P}_1}^2 & \xrightarrow{j_f} & \mathcal{O}_{\mathbb{P}_1}^4 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T_{\mathbb{P}_1} & \longrightarrow & f^* T_{\mathbb{P}_3} & \longrightarrow & N_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

then we see that N_f is naturally isomorphic to $\operatorname{coker}(j_f)$. From this observation the lemma readily follows.

Proof of Theorem 5.1. (After an idea of Hulek.) The condition $g(n) \circ j_f = 0$ is equivalent to a system of linear conditions on the coefficients of f. So, since $S_{n,a+1} \cup \ldots$ is closed in S_n, the fibres of G are open subsets of linear spaces in $\mathbb{P}_{4n+3}$. Furthermore, since the map associating to f the linear conditions $g(n) \circ j_f = 0$ on g is regular, the following elementary consequence of "Cramer's rule" for solving systems of linear equations over a field completes the proof.

Lemma 5.3. *Let* $\mathbb{P}_{rs-1}$ *be the projective space, associated to the space of* $r \times s$-*matrices* M *with coefficients in a field* $\mathfrak{k}$, *with* $r \leqq s$, *considered as homomorphisms from* $\mathfrak{k}^r$ *into* $\mathfrak{k}^s$. *Let* $U \subset \mathbb{P}_{rs-1}$ *be the variety of matrices of rank exactly* $r-1$. *Then there is a regular map* $\varphi: U \to \mathbb{P}_{r-1}$, *such that* $\varphi(M) = \ker M$ *for all* $M \in U$.

6. The Generic Family

Let $R'_{n-a-1} \subset R_{n-a-1}$ consist of those $g \in R_{n-a-1}$, which satisfy the following conditions.

(i) $k_0(g) = 0$;

(ii) $g(\mathbb{P}_1)$ is non-planar;

(iii) $N_g \cong \mathcal{O}^2_{\mathbb{P}_1}(2n-2a-3)$.

All these conditions are open and somewhere satisfied. As to (ii) this follows from $a \leqq n-4$, whereas (iii) follows from the remark after Proposition 4.1 together with the main result of [3].

We now pick any $g_0 \in R'_{n-a-1}$ and consider the fibre $G^{-1}(g_0)$. For every $f \in G^{-1}(g_0)$ we know from Proposition 3.2 that $f(\mathbb{P}_1)$ lies on the tangent surface T of the dual $g_0^\vee$ of g_0. Moreover, f can be lifted in a unique way to a morphism $\tilde{f}: \mathbb{P}_1 \to \tilde{T}$, where $\tilde{T}$ is the normalisation of T, which is smooth by [3]. The image $\tilde{f}(\mathbb{P}_1)$ then becomes a section. Conversely, since the parametrisation is completely fixed by that of g, this section determines $\tilde{f}$ and f. In this way the curves in $G^{-1}(g_0)$ correspond to a certain system of sections on $\tilde{T}$. We want to show that this system is an open subset of a linear system of divisors and we want to show that it has the right dimension.

First of all we want to observe that $\tilde{T}$ is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. This is a consequence of

Lemma 6.1. *If* $g: \mathbb{P}_1 \to \mathbb{P}_3^\vee$ *is smooth with non-planar image, and* $\tilde{T}$ *denotes the normalisation of the tangent surface of* $g^\vee: \mathbb{P}_1 \to \mathbb{P}_3$, *then* $\tilde{T} \cong \mathbb{P}(N_g)$.

Proof. It follows from duality theory ([4]) that the Gauss maps of g and $g^\vee$ are dual under the canonical duality between the Grassmannians of lines in $\mathbb{P}_3^\vee$ and $\mathbb{P}_3$. Thus the points on the tangent line to $g^\vee(\mathbb{P}_1)$ at $g^\vee(t)$ may be identified with the planes containing the tangent line to $g(\mathbb{P}_1)$ at $g(t)$; but these planes may further be identified with the lines in the normal bundle to g at t, and the lemma follows.

Now let $\tilde{H}$ be the hyperplane class on $\mathbb{P}_3$, pulled back to $\tilde{T}$, and let $\tilde{F}$ be the class of the fibres (corresponding to the tangent to $g^\vee$) on $\tilde{T}$. Since $\tilde{H} \cdot \tilde{f}(\mathbb{P}_1) = n$, we see that the class of $\tilde{f}(\mathbb{P}_1)$ is $\tilde{H} + (n - \tilde{H}^2)\tilde{F}$. Using the Plücker formulas we get $\tilde{H}^2 = \deg T = d_1(g^\vee) = d_1(g) = 2(n-a-1) - 2 = 2n - 2a - 4$, since $k_0(g) = 0$. So $\tilde{f}(\mathbb{P}_1) \sim \tilde{H} - (n-2a-4)\tilde{F}$, and $(\tilde{f}(\mathbb{P}_1))^2 = 2a+4 > 0$. Consequently, $f(\mathbb{P}_1)$ is very ample, since we are on $\mathbb{P}_1 \times \mathbb{P}_1$. Furthermore, $\dim |\tilde{f}(\mathbb{P}_1)| = (\tilde{f}(\mathbb{P}_1))^2 + 1 = 2a + 5$, as follows from

Lemma 6.2. *Let* X *be a smooth algebraic surface with* $H^1(\mathcal{O}_X) = 0$ *and let* C *be a smooth rational curve on* X. *Then we have* $\dim |C| = \max(C^2 + 1, 0)$.

Proof. Taking the cohomology of the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|C \to 0$$

we see that $\dim|C| = h^0(\mathcal{O}_X(C)) - 1 = h^0(\mathcal{O}_X(C)|C)$, which is C^2+1 if $C^2 \geqq 0$ and 0 if $C^2 < 0$.

Next we recall a lemma, proved in our first paper ([3], Proposition 2), namely

Lemma 6.3. *Let $h: X \to Y$ be a finite birational map between surfaces, with X smooth, and let C be a very ample divisor on X. Then, $h(C')$ is smooth for almost all C', which are linearly equivalent to C.*

Combining all this we conclude that $G^{-1}(g_0)$ has dimension exactly $2a+5$. Since $\dim R'_{n-a-1} = 4n-4a-1$, it follows that $S'_{n,a} = G^{-1}(R'_{n-a-1})$ has dimension $4n-2a+4$, as required. It remains to show that $S'_{n,a}$ is irreducible. But if X is any component of $S'_{n,a}$, then the closure of X in $S_{n,a}$ is a component of $S_{n,a}$, so $\dim X \geqq 4n-2a+4 = \dim S'_{n,a}$ by Proposition 4.1. It follows that $G(X)$ is dense in R'_{n-a-1}, and that for $g \in G(X)$ we have that $X \cap G^{-1}(g) = S'_{n,a} \cap G^{-1}(g)$. If now Y were another component, then the same would hold for Y, so $\dim(Y \cap X) = \dim X$, which implies $X = Y$. Thus $S'_{n,a}$ is irreducible as claimed.

To show that $S'_{n,a}$ is rational we note that, by Theorem 5.1, the fibres of $G|S'_{n,a}$ are open subsets of linear subspaces in the ambient $\mathbb{P}_{4n+3}$. So $S'_{n,a}$ is birationally equivalent to the product of R'_{n-a-1} and a linear space (since the fibres of $G|S'_{n,a}$ are open subsets of *linear* subspaces in $\mathbb{P}_{4n+3}$, there exist over an open subset of R'_{n-a-1} enough sections), hence $S'_{n,a}$ is rational.

7. Four Types of Smaller Families of Given Degree and Normal Bundle

In this final section we shall show that the closure of $S'_{n,a}$ in $S_{n,a}$ is $S_{n,a}$ itself, thus completing the proof of our main result. To this purpose it is sufficient to show that there is no component X of $S_{n,a}$, such that for a general $f \in X$, the map $g = G(f)$ satisfies one of the following conditions:

(i) $g(\mathbb{P}_1)$ is planar;

(ii) $g(\mathbb{P}_1)$ is not planar and g is not birational;

(iii) $g(\mathbb{P}_1)$ is not planar, g is birational, $k_0(g)=0$, but N_g is not balanced;

(iv) $g(\mathbb{P}_1)$ is not planar, g is birational and $k_0(g) \geqq 1$.

Our strategy will be quite simple: analysing the dimension of $G(X)$ and the dimension of a general fibre $G^{-1}(G(f))$ we show that such a component would have a dimension $< 2n-2a+1$, which is impossible by Proposition 4.1.

Since the space of maps $\mathbb{P}_1 \to \mathbb{P}_2$ of degree $n-a-1$ has dimension $3(n-a)-1$, and the family of planes in $\mathbb{P}_3^\vee$ is 3-dimensional, we have that $\dim G(X) \leqq 3n-3a+2$.

Next we fix any $g \in G(X)$ and show that $\dim G^{-1}(g) \leqq n+2$, which will be sufficient.

(i) $g(\mathbb{P}_1)$ is planar

Let $C=f(\mathbb{P}_1)$. The image $g(\mathbb{P}_1)$ can't be contained in a line, otherwise all tangents to C would meet a same L, and by projecting C from a general point of L onto a plane we would obtain a curve of degree $\geqq 2$ (C itself is non-planar), all tangents of which would pass through a fixed point, and this is impossible. Thus $g(\mathbb{P}_1)$ spans a plane $V\subset\mathbb{P}_3^\vee$.

Let $S\subset\mathbb{P}_3$ be the cone whose tangent planes are dual to the points of $g(\mathbb{P}_1)$, so that the vertex s of S is the point dual to V. By Proposition 3.1 we have that $C\subset S$.

Then let B be the intersection of S with a general plane. Since C projects from s onto B, we see that B is rational. Let as before G be the Grassmannian of lines in $\mathbb{P}_3$. We denote by $h\colon \tilde{B}\to B$ the normalisation of B, and by $\ell\colon \tilde{B}\to G$ the map which attaches to $\tilde{b}\in\tilde{B}$ the line joining $h(\tilde{b})$ with s. The surface $\tilde{S}=\{(\tilde{b},p)\in\tilde{B}\times\mathbb{P}_3\,|\,p\in\ell(b)\}$ is the pull-back of the universal projective subbundle on G. The obvious map $\pi\colon\tilde{S}\to S$ is the desingularisation of S, and $\tilde{S}$ is a rational ruled surface. Since C can't be contained in the singular locus of S, we may lift C to a smooth curve $\tilde{C}$ on $\tilde{S}$. If we write $\tilde{C}_0$ for $\pi^{-1}(S)$ and $\tilde{F}$ for a fibre of $\tilde{S}$, then $\tilde{C}_0$ is a section on $\tilde{S}$ and $-e=\tilde{C}_0^2\leqq -1$, for $\tilde{C}_0$ can be blown down in $\tilde{S}$. Since the pull-back of a general hyperplane section of S to $\tilde{S}$ does not meet $\tilde{C}_0$, and meets $\tilde{F}$ once, we have $\tilde{C}\sim a\tilde{C}_0+n\tilde{F}$ for some $a\in\mathbb{Z}$. Since $\tilde{C}$ is irreducible, $a\geqq 1$, and since C is smooth we must have $\tilde{C}_0\cdot\tilde{C}\leqq 1$, that is $ae\leqq n-1$.

We shall now show that $a=1$. Since $\tilde{C}$ is rational, the adjunction formula gives

$$2=\tilde{C}(\tilde{C}+K_{\tilde{S}})$$

where $K_{\tilde{S}}=-2\tilde{C}_0-(e-2)\tilde{F}$ is the canonical class of $\tilde{S}$. Using $\tilde{C}_0^2=-e$, $\tilde{C}_0\tilde{F}=1$ and $\tilde{F}^2=0$ this becomes

$$(a-1)(2n-ae+2)=0.$$

If $a>1$, then $ae=2n+2$, contradicting the previous inequality. So $a=1$, as claimed.

It is now clear that $G^{-1}(g)$ may be identified with an open subset of the linear system $|\tilde{C}|=|\tilde{C}_0+n\tilde{F}|$ on $\tilde{S}$. Putting $\mathscr{L}=[\tilde{C}]$, we see that it is sufficient to show that $h^0(\mathscr{L})\leqq n+3$. Using the exact sequence

$$0\to\mathcal{O}_{\tilde{S}}\to\mathscr{L}\to\mathscr{L}\otimes\mathcal{O}_{\tilde{C}}\to 0$$

we deduce that $h^0(\mathscr{L})=1+h^0(\mathscr{L}\otimes\mathcal{O}_{\tilde{C}})$. Since $\tilde{C}\cong\mathbb{P}_1$, and the degree of $\mathscr{L}\otimes\mathcal{O}_{\tilde{C}}$ is $\tilde{C}^2=2n-e$, we have $h^0(\mathscr{L})\leqq 2+2n-e\leqq n+3$, as required.

(ii) $g(\mathbb{P}_1)$ is not planar and g is not birational

Suppose $\deg(g)=k$, with $k\geqq 2$. Then the degree of $g(\mathbb{P}_1)$ is $(n-a+1)/k$ and since $g(\mathbb{P}_1)$ is not planar we may suppose $k\leqq(n-a-1)/3$. In particular, since $a\geqq 1$ and $k\geqq 2$ we have $n\geqq 8$.

By way of the map $(g_1,g_2)\to g_2g_1$, the variety W of k-to-1 maps $\mathbb{P}_1\to\mathbb{P}_3^\vee$, whose image is a curve of degree $(n-a-1)/k$, is visibly an image of the variety of pairs (g_1, g_2), with $g_1 : \mathbb{P}_1 \to \mathbb{P}_1$ a map of degree k, and $g_2 : \mathbb{P}_1 \to \mathbb{P}_3^\vee$ a map of degree $(n-a-1)/k$. If $\sigma \in \mathrm{Aut}(\mathbb{P}_1)$, then $(\sigma g_1, g_2 \sigma^{-1})$ give the same map as (g_1, g_2). From this we compute

$$\dim G(X) \leqq \dim W = [2(k+1)-1] + \left[4\left(\frac{n-a-1}{k}+1\right)-1\right]-3$$
$$= 2k + 4(n-a-1)/k + 1.$$

Next, we fix g and consider the fibre $G^{-1}(g)$, which, by Proposition 3.2, may be thought of as a set of smooth rational parametrised curves f: $\mathbb{P}_1 \to \mathbb{P}_3$, such that $C = f(\mathbb{P}_1)$ lies on the tangent surface $T(g^\vee)$, where the parametrization f of C is compatible in an obvious way with that of $g^\vee$ and g. As in Sect. 5, we can lift each f to a map $\tilde{f}$: $\mathbb{P}_1 \to \tilde{T}$, where $\tilde{T}$ is the smooth normalisation of T. Then $\tilde{f}(\mathbb{P}_1)$ is a smooth rational curve on $\tilde{T}$, meeting the fibres in more than one point. From this, combined with the adjunction formula and the rationality of $\tilde{f}(\mathbb{P}_1)$, it follows that $\tilde{T}$ must be $\Sigma_0 = \mathbb{P}_1 \times \mathbb{P}_1$, and that $\tilde{f}(\mathbb{P}_1)$ meets "the other fibres" in one point. Consequently, $\dim|\tilde{f}(\mathbb{P}_1)| = k+2$.

It now follows that

$$\dim X \leqq [2k + 4(n-a-1)/k + 1] + k + 2,$$

and it remains to show that this number is $< 4n - 2a + 4$. Multiplying by k and subtracting the left hand side from the right, we must show that

$$0 < -3k^2 + (4n - 2a + 1)k - 4n + 4a + 4.$$

The coefficient of a is $4 - 2k$, which is $\leqq 0$, so it suffices to check this for the maximum value of a, which is $n-4$. Making this replacement, it suffices to show that

$$0 < -3k^2 + (2n+9)k - 12$$

for $2 \leqq k \leqq \frac{n-2}{3}$. Since the given expression is quadratic in k, with negative leading term, it suffices to check the inequality for the extreme values of k, for which it becomes

$$0 < 4n - 6 \quad \text{and} \quad 0 < n^2 + 9n - 58$$

respectively, and these are certainly satisfied if, as we have assumed, $n \geqq 8$.

(iii) $g(\mathbb{P}_1)$ is non-planar, g is birational, $k_0(g) = 0$, but N_g is not balanced

In this case we have that $\dim G(X) < 4n - 4a - 1$. Furthermore, we find in the same way as in the preceding section that the dimension of any fibre of G is at most $2a + 5$, whence the result.

(iv) $g(\mathbb{P}_1)$ is non-planar, g is birational and $k_0(g) \geqq 1$

We shall show below that the dimension of a general fibre is at most $2a+k_0+5$. On the other hand, we know from Theorem 2.2 that the space of maps $g: \mathbb{P}_1 \to \mathbb{P}_3^{\vee}$ of degree $n-a-1$ having $k_0 = k_0(g)$ cusps is either empty or of dimension $4n-4a-2k_0-1$, so using the announced bound for $\dim G^{-1}(g)$ we get the desired result.

It remains to prove that $\dim G^{-1}(g)$ is at most $2a+k_0+5$. This again can be done by the method of Sect. 6. The only difference is that here $\tilde{H}^2 = 2n-2a-k_0-4$, leading to $\tilde{C}^2 = 2a+k_0+4$ and hence $\dim G^{-1}(g) \leqq 1+\tilde{C}^2 = 2a+k_0+5$.

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