REPORT ON NORMAL BUNDLES OF CURVES IN $\mathbb{P}_3$

by

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There has recently been a little flurry of activity devoted to an apparently new corner of the hoary subject of smooth curves in projective 3-space, namely the study of the normal bundles of such curves. The subject is still embryonic, but there are many concrete and interesting problems. I wish here to surrey the results obtained so far, which treat curves of degree ≤ 5 and rational curves.

Throughout curves will be non singular, and everything will take place over $\mathbb{C}$, though this last is probably inessentiel.

A good deal of the activity in this subject seems to have been triggered by a remark of Grauert, to the effect that one had (18 months ago) no single example of a curve in $\mathbb{P}_3$ whose normal bundle was indecomposable.

What would be the simplest possible such example ? Smooth curves of degrees 1, 2, or 3 are rational or complete intersections, and every vector bundle over $\mathbb{P}_1$, decomposes. The only non-rational smooth curves of degree 4 are the elliptic quartics obtained as the intersection of two quadric surfaces, which again have decomposable normal bundles because they are complete intersection So one must go to degree 5 for the first interesting examples.

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There are two sorts of non-rational smooth curves of degree 5 in $\mathbb{P}_3$ (by Castel-nuovo or by R-R directly) : the <u>elliptic curves and curves of genus</u> 2 (hyperelliptic curves). Now, h^o (5 points on an elliptic curve) = 5 - so there is a natural embedding in $\mathbb{P}_4$ - and the elliptic quintics are projections of this. On the other hand the hyperelliptic curves are not projections, and in fact they are arithmetically normal ; i.e. $\sum H_o(\mathcal{O}_C(n))$ is integrally closed and we will treat these hyperelliptic curves first.

$$R = k\,[X_0,X_1,X_2,X_3] \twoheadrightarrow \sum H_o(\mathcal{O}_C(n)) \quad \text{is an epimorphism.}$$

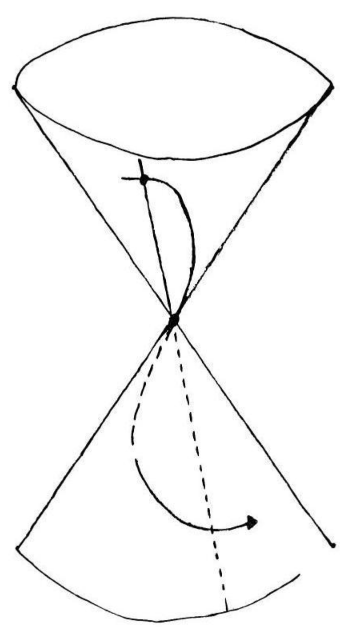
Now

$$h^o(\mathcal{O}_C(2)) = h^o(\mathcal{O}_C(2)) - h^1(\mathcal{O}_C(-2) \otimes K)$$

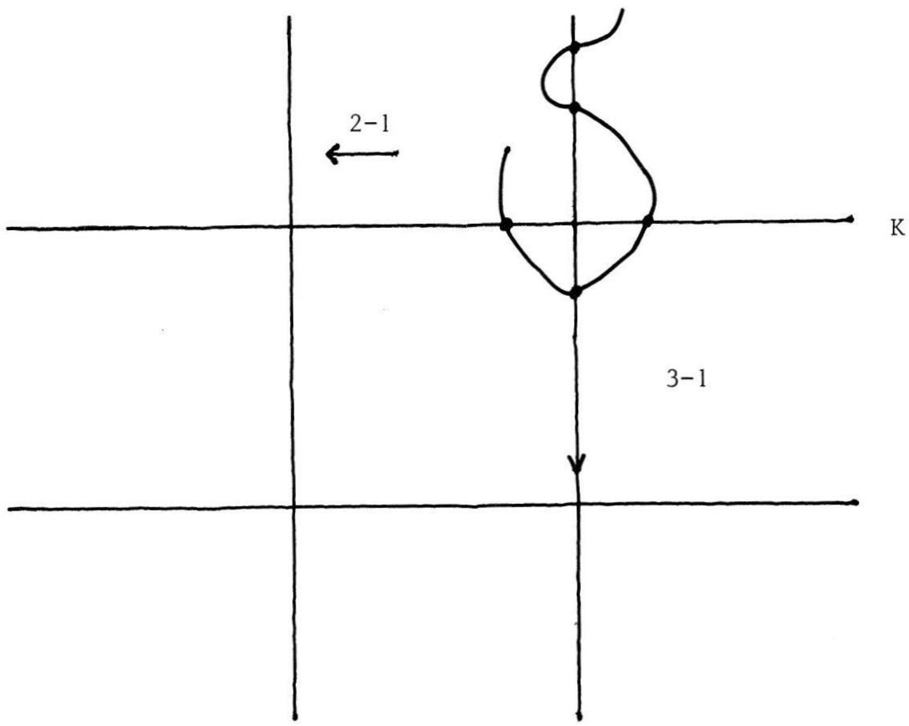
$$= 10 + 1 - 2 = 9.$$

So such a curve lies on a unique, singular or non singular, quadric,

The distinction between hyperelliptic curves an smooth and an singular conics is fundamental, as one can see in another way : let C be <u>any</u> genus 2 curve, K (of degree 2) its canonical series.If D is a divisor of degree 5, then h^o D = 4 and $|D| = \mathbb{P}_3$. Thus of the 5 points of D we may choose 3 to come from 2K ; if a 4^{th} does too, then the image of C in $\mathbb{P}_3$ lies on a quadric cone :



(a hyperplane section of the cone through the vertex is 2 lines and meets the curve in 5 points). If only 3 points of D are in 2K , then the image of C in $\mathbb{P}_3$ lies on a nonsingular quadric, isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, as a divisor of class (3,2) :



<u>Theorem</u> (Van de Ven -Comptes Rendus- Spring 1979). <u>If</u> C <u>is nonsingular curve of degree</u> 5 <u>and genus</u> 2 <u>in</u> $\mathbb{P}_3$, <u>then</u> N_C <u>decomposes if and only if</u> C <u>lies on a singular quadric</u>.

In the situation of the Theorem, it is easy to write equations for C , so one can "see" the normal bundle of C very directly as $(I_C/I_C^2)^*$; in fact, if C lies on a non singular quadric

$X_0X_3-X_1X_2$, then its equations are the 2×2 minors of a matrix of the form :

$$\begin{pmatrix} X_0 & X_1 & Q_1 \\ X_2 & X_3 & Q_2 \end{pmatrix}$$

while if C lies on a singular quadric $X_0X_2-X_1^2$ then its equations are the minors of :

$$\begin{pmatrix} X_0 & X_1 & Q_1 \\ X_1 & X_2 & Q_2 \end{pmatrix}$$

where in each case Q_1,Q_2 are quadratic forms ; and if Q_1,Q_2 are chosen such that the curve defined by the minors of one of these matrices is smooth, then it is hyperelliptic of the correct type. But I do not know how to recover Van de Ven's result by this method.

What about the elliptic case ? This was first treated incorrectly, but eventually the beautiful story became clear in a paper of Ellingsrud and Laksov ; as we have mentioned, elliptic quintics in $\mathbb{P}_3$ are obtained by projection of "the" elliptic quintic C in $\mathbb{P}_4$ from a point $p\in\mathbb{P}_4$ (The notion of "the" elliptic quintic in $\mathbb{P}_4$ is justified by the fact that any two divisor classes of the same degree on an elliptic curve differ by an automorphism of the curve). So for each $p\in\mathbb{P}_4$, not in the secant locus Sec (C) of C, we let C_p be the projection of C from p and we write N_p for its normal bundle. A Chern class computation shows that if N_p decomposes, then it does so as $M_o\otimes\mathcal{L}^2\oplus M_o^{-1}\otimes\mathcal{L}^2$, where M_o is a line bundle of degree 0 and $\mathcal{L}$ is the bundle corresponding to the hyperplane section of C in $\mathbb{P}_4$. The main result is :

<u>Theorem.</u> (Ellingsrud-Laksov-Comptes Rendus-to appear). <u>For each line bundle</u> M_o <u>of degree</u> 0 <u>on</u> C <u>there is a quintic hypersurface</u> $H_{M_o}\subset\mathbb{P}_4$, <u>such that</u> :

1) $H_{M_o}=H_{M_o^{-1}}$

2) $\bigcup_{M_o} H_{M_o}=\mathbb{P}_4$; $H_{M_o}\cap H_{M'_o}\subset \mathrm{Sec}(C)$ if $M_o\neq M_o'^{\pm 1}$

3) $H_{M_o}\subset \mathrm{Sec}\ C \iff M_o=\mathcal{O}$, <u>in which case</u> $H_{M_o}=\mathrm{Sec}\ C$.

4) <u>For any</u> $p\in\mathbb{P}_4-\mathrm{Sec}\ C$ $p\in H_{M_o}$ <u>iff there is an exact sequence of vector bundles</u>.

$$0\longrightarrow M_o^{-1}\otimes\mathcal{L}^2\longrightarrow N_p\longrightarrow M_o\otimes\mathcal{L}^2\longrightarrow 0$$

5) <u>If</u> $M_o^2 \neq \mathcal{O}$, <u>the sequence above splits</u> ; <u>if</u> $M_o^2 = \mathcal{O}$, <u>then for suitable</u> (conjecturally : for all) $p \in H_{M_o}$ - Sec C, N_p <u>is indecomposable.</u>

A second proof of this result has been found by Van de Ven and my self ; its key point is to exploit the fact that all the equations of C in $\mathbb{P}_4$ are of the same degree (2), and all the relations on them are generated by linear relations. In fact, the homogeneous ideal of C is generated by the 4×4 Pfaffians of a matric of the form :

$$\begin{pmatrix} 0 & 0 & X_0 & X_1 & X_3 \\ 0 & 0 & X_1 & X_2 & X_4 \\ -X_0 & -X_1 & 0 & A & B \\ -X_1 & -X_2 & -A & 0 & C \\ -X_3 & -X_4 & -B & -C & 0 \end{pmatrix}$$

where A, B and C are linear forms depending on the embedding chosen for C in $\mathbb{P}_4$.

So far, because of our quest for indecomposable normal bundles, we have left out what ought to the first and simplest case of all, the case of rational curves. Last fall, Van de Ven and I set out to plug this gap, and I now want to describe some of the results we have obtained.

First of all, any rank 2 bundle on $\mathbb{P}_1$ is of the form $B = \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b)$ for some integers a and b. The numbers a and b are analytic but not topological invariants of the bundle, but $a + b = \deg(\det B)$ is, and it is correspondingly easier to compute.

Every rational curve of degree n in $\mathbb{P}_3$ is the projection from some $p \cong \mathbb{P}_{n-4} \subset \mathbb{P}_n$ of "the rational normal n-ic" C in $\mathbb{P}_n$ (up to ambient automorphisms), so we may parametrize every thing by projection centers p, and write C_p for a rational n-ic in $\mathbb{P}_3$, and N_p for its normal bundle. The normal bundle N_p is obtained from N_C by factoring out a sub-bundle isomorphic to $\overset{n-4}{\oplus} \mathcal{O}_{\mathbb{P}_n}(1)\big|_C$ induced by P, and one computes easily from this that if $N_p = \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b)$, then $a + b = 4n-2$.

From general principles, one now expects the general C_p to have $N_p = \mathcal{O}_{\mathbb{P}_1}(2n-1) \oplus \mathcal{O}_{\mathbb{P}_1}(2n-1)$, a "balanced bundle", and this is in fact the case. (Example : the curve parametrized by $(s^n, s^{n-1}t, st^{n-1}, t^n)$). So the first question is : how unbalanced can N_p be ?

With the above notation, if C_p is smooth and $n > 3$, then :

Proposition : $n+3 \leq a,b \leq 3n-5$.

One now can construct examples to show that every a,b subject to the above conditions actually occurs as the normal bundle of a smooth rational curve. One way to evaluate the computations is to use the following result, which gives a geometric description of a and b :

Theorem : Let $C \subset \mathbb{P}_3$ be a smooth, rational curve of degree n, $T(C)$ its tangent surface (the union of its tangent lines, which is a surface of degree $2n-2$). Let L be a line meeting $T(C)$ transversely and consider $\Gamma \subset C \times L$,

$$\Gamma = \{(C, p) \mid T_C \ni p\} .$$

Then $N_C = \mathcal{O}((2n-1)+a) \oplus \mathcal{O}((2n-1)-a)$ iff Γ lies on the graph of a map $C \longrightarrow L$ of degree $(n-1)-a$.

Using this we can exhibit an interesting example of a maximally unbalanced curve :

Example. Let D be the twisted cubic, $T(D)$ its tangent surface. $T(D)$ is the projection of a ruled surface S of degree 4 in $\mathbb{P}_5$ (S is $\mathbb{P}_1 \times \mathbb{P}_1$, embedded linearly in the first factor and quadratically in the second). Let H be the hyperplane section on S, R the ruling. The divisor $H + (n-4)R$ is very ample for all $n \geq 4$, and is a rational curve (since intersection with rulings induces an isomorphism with $\mathbb{P}_1$). The general divisor in this class will project to a smooth rational n-ic C in $\mathbb{P}_3$ lying on $T(D)$ and meeting each tangent line once . We claim that $N_C = \mathcal{O}_{\mathbb{P}_1}(n+3) \oplus \mathcal{O}_{\mathbb{P}_1}(3n-5)$. To check this, let L be a general line in $\mathbb{P}_3$, and consider the map $\varphi : C \longrightarrow L$ obtained by carrying $z \in C$ to $T_{T(D),z} \cap L \in L$. If, for some $z \in C$, $T_{C,z} \cap L = \bar{z}$, then clearly $\varphi(z) = \bar{z}$. By the Theorem, it will be enough to show that φ is a map if degree 3 ; or equivalently, that the number of points $z \in C$ for which $T_{T(D),z}$ contains a given point, is 3. But, as a local computation shows, $T_{T(D),z}$ is just the osculating plane of D at the point of D whose tangent passes through z. Since the dual curve to D has degree 3, we are done. ||

Added in Proof : After the above was written, I have became aware of papers by Ghione and Sacchiero, and Sacchiero (Preprints, Univ. of Ferrara), giving bounds on the normal bundles of rational curves with ordinary singularities in $\mathbb{P}_n$ (known to Van de Ven and me for $n=3$), and some (singular) examples of such curves.