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On the Normal Bundles of Smooth Rational Space Curves

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I. Introduction

In this note we consider smooth rational curves C of degree n in three-dimensional projective space $\mathbb{P}_3$ (over a closed field of characteristic 0). To avoid trivial exceptions we shall always assume that $n \geqq 4$ (this does not hold however for certain auxiliary curves we shall consider). Let $N = N_C$ be the normal bundle of C in $\mathbb{P}_3$. Since $\deg c_1(\mathbb{P}_3) = 4$, and $\deg c_1(\mathbb{P}_1) = 2$, we have that $\deg c_1(N) = 4n - 2$. By a well-known theorem of Grothendieck the bundle N is a direct sum of two line bundles. Hence $N \simeq \mathcal{O}_C(2n-1-a) \oplus \mathcal{O}_C(2n-1+a)$ for some non-negative $a = a(C)$, which is uniquely determined by C. The question we would like to answer is an obvious one: which values of a occur? We shall show (Theorem 4 below) that a value a occurs if and only if $0 \leqq a \leqq n-4$.

Since for every smooth space curve the normal bundle is generated by global sections we have in any case that $N_C \simeq \mathcal{O}_C(m_1) \oplus \mathcal{O}_C(m_2)$, with $m_1, m_2 \geqq 0$, therefore $H^1(C, N) = 0$. It follows [K, p. 150] that C represents a smooth point on the Chow variety $\mathrm{Ch}(3, 1, n)$ of effective cycles of dimension 1 and degree n in $\mathbb{P}_3$. Since the set of all smooth rational curves with a fixed degree is obviously connected, we see that the smooth C's represent a smooth, irreducible, $4n$-dimensional (Zariski-)open subset S of $\mathrm{Ch}(3, 1, n)$.

In a forthcoming paper [E–V] we shall prove the following

Theorem. *Let S be the smooth, irreducible $4n$-dimensional (Zariski-) open subset of $\mathrm{Ch}(3, 1, n)$, which parametrises the smooth rational space curves of degree n in $\mathbb{P}_3$. Then there exists a stratification of S by non-empty, locally-closed subsets S_i, $0 \leqq i \leqq n-4$ $\left(\text{i.e. } S = \bigcup_{i=0}^{n-4} S_i \text{ and } S_{i+1} \subset \bar{S}_i\right)$, such that $N_C \simeq \mathcal{O}_C(2n-1-i) \oplus \mathcal{O}_C(2n-1+i)$ if and only if the point representing C is contained in S_i. For $1 \leqq i \leqq n-4$, the set S_i is irreducible of dimension $4n-2i+1$.*

In particular this implies that for a general curve C the normal bundle N_C is balanced, that is, $N_C \simeq \mathcal{O}_C(2n-1) \oplus \mathcal{O}_C(2n-1)$ (see also [Ha, Corollary 3]).

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After proving Theorem 4 in Sect. III we turn in Sect. IV to some special topics. If C is a smooth rational curve of degree n, then we prove that C lies on a smooth quadric if and only if the tangent bundle of $\mathbb{P}_3$, restricted to C is isomorphic to $\mathcal{O}_C(2n-2)\oplus\mathcal{O}_C(n+1)\oplus\mathcal{O}_C(n+1)$. On the other hand, if C lies on a smooth quadric, the normal bundle of C is isomorphic to $\mathcal{O}_C(2n-1)\oplus\mathcal{O}_C(2n-1)$, the same as if C were a general rational curve. (This last fact was also noted by Harris [Ha] as a corollary of a more general result. It will also appear, with a different proof, in the paper [Hu] of Hulek.)

After a first draft of this paper was written, we obtained a preprint of the paper [G-S] by Ghione and Sacchiero, in which the normal bundles of curves with at most ordinary singularities are studied. Ghione and Sacchiero prove a theorem similar to our Theorem 4, except that they allow curves with ordinary singularities and correspondingly must weaken the bound on a to $0\leqq a\leqq n-3$. Since they show that curves with $a=n-3$ lie on quadric cones, they also get the bound $0\leqq a\leqq n-4$ for smooth curves. By contrast, their proof, by example, for the existence of singular curves with all possible values of a, does not, as far as we see, give a systematic production of smooth curves of the given types. The main part of our paper is to give a method for systematically constructing such smooth examples.

II. Some Preliminaries

1. Some Classical Formulae for Space Curves

Let D be a curve in $\mathbb{P}_3$, which is not contained in a plane, let $\tilde{D}$ be the normalisation of D, and $h\colon \tilde{D}\to D$ the canonical projection. Given any point $p\in\tilde{D}$, there exist affine coordinates (ξ_1,ξ_2,ξ_3) in a neighborhood of $h(p)=(0,0,0)$, such that the branch of D, determined by p has a formal parametrisation

$$\begin{aligned}\xi_1&=t^{l_0(p)+1}+(\text{terms of higher order})\\ \xi_2&=t^{l_0(p)+l_1(p)+2}+(\text{terms of higher order})\\ \xi_3&=t^{l_0(p)+l_1(p)+l_2(p)+3}+(\text{terms of higher order})\end{aligned}$$

in a neighborhood of $h(p)$ $(l_0,l_1,l_2\geqq 0)$. The numbers l_0, l_1, and l_2 are independent of the parametrisation thus chosen. We put $l_0(D)=\sum_{p\in\tilde{D}} l_0(p)$ and $l_1(D)=\sum_{p\in\tilde{D}} l_1(p)$. Then there are (compare [P, 3.2]) classical formulae for the degree $\deg(F_D)$ of the tangent (developable) surface F_D of D (that is the surface formed by the tangent lines to D) as well as for the class $r(D)$ of D (the number of osculation planes, passing through a general point of $\mathbb{P}_3$):

$$\begin{aligned}\deg(F_D)&=2\deg(D)+2\Pi(D)-2-l_0(D)\\ r(D)&=3(\deg D+2\Pi(D)-2)-2l_0(D)-l_1(D).\end{aligned}$$

Here $\Pi(D)$ denotes the geometric genus of D (the genus of $\tilde{D}$).

2. *Rational Curves with Certain l_0, l_1*

Let A_{norm} be a smooth rational curve of degree d in $\mathbb{P}_d$, $d \geqq 4$. We denote by F the tangent surface of A_{norm} and by G the osculation variety of A_{norm} (the union of the osculating two-planes). We have $F \subset G$. If P is a $(d-4)$-dimensional linear subspace of $\mathbb{P}_d$ with $P \cap A_{\mathrm{norm}} = \emptyset$, then A_{norm} projects from P to a rational curve A, of degree d, which spans the $\mathbb{P}_3$ of all $\mathbb{P}_{d-3}$'s passing through P. The following elementary facts can be verified immediately:

(i) if $P \cap G$ (and hence $P \cap F) = \emptyset$, then A is a rational curve with $l_0(A) = l_1(A) = 0$;

(ii) if $P \cap F = \emptyset$ and $P \cap G$ consists of one point x, which is smooth on G, such that P and the tangent space to G at x are independent, then $l_0(A) = 0$, $l_1(A) = 1$;

(iii) if $P \cap G$ consists of one point x, contained in F, such that x is smooth on F, and such that P and the tangent space to F at x are independent, then $l_0(A) = 1$, $l_1(A) = 0$.

Since $l_0(A) = l_1(A) = 0$ also holds for a smooth cubic in $\mathbb{P}_3$, we have

Proposition 1. (i) *For every $d \geqq 3$ there are rational curves A of degree d, spanning $\mathbb{P}_3$, with $l_0(A) = l_1(A) = 0$;*

(ii) *for every $d \geqq 4$ there are rational curves A of degree d, spanning $\mathbb{P}_3$, with $l_0(A) = 0$, $l_1(A) = 1$, and also with $l_0(A) = 1$, $l_1(A) = 0$.*

3. *A Property of Very Ample Divisors on a Surface*

Proposition 2. *Let X be a smooth surface, Y a smooth variety and $f\colon X \to Y$ a finite map, which maps X birationally onto its image. If D is a very ample divisor on X, and $E \in |D|$ sufficiently general, then $f(E)$ is smooth (and $f|E$ an isomorphism).*

Proof. We have to show the following: if $E \in |D|$ is sufficiently general, then (i) $f|E$ is of maximal rank in every point of E and (ii) $f|E$ is one-to-one.

As to (i) the divisor of E first of all has to avoid all points on X, where f has rank 0. This is easily possible since there is only a finite number of such points. Secondly, E must not be tangent to the kernel of $\mathrm{d}f$ at any point of X where f has rank 1. But the space of E's which is tangent to the kernel of $\mathrm{d}f$ in one of these points has codimension 2 in $|D|$; and since the union of all these points consists of finitely many curves minus a finite number of points (where f has rank 0), we see that a general E is not tangent to any of the kernels mentioned before.

To prove (ii), let Z be the closure of the set of points $z \in f(X)$, for which $f^{-1}(z)$ consists of at least two points. Since f is birational, Z is a proper subset of $f(X)$, and therefore consists of finitely many points and curves. A general E certainly does not meet the inverse image of the union of the points. Hence it is enough to show the following: given an irreducible curve $F \subset f(X)$, such that $f\colon f^{-1}(F) \to F$ is generically k-to-1, with $k \geqq 2$, then the general curve E does not contain any two different points on the same fibre of $f|f^{-1}(F)$. But this is clear, for the space of elements in $|D|$, containing two different points of such a fibre has codimension two at least.

4. *A Property of the Tangent Surface*

Proposition 3. *Let C be a space curve (possibly singular), which is not contained in any plane. If F is the tangent surface of C, and $\tilde{F}$ its desingularisation, then the projection from $\tilde{F}$ onto F is a finite map.*

Proof. Let $\tilde{C}$ be the normalisation of C. We can extend the Gauss map, which is a priori defined in all but a finite number of points on $\tilde{C}$, to a regular map $g\colon \tilde{C}\to G$, where G is the Grassmann variety of lines in $\mathbb{P}_3$. Let E be the pull back on $\tilde{C}$ of the $\mathbb{P}_1$-bundle, associated to the universal subbundle on G. The surface E is contained in $G\times\mathbb{P}_3$ in a natural way, and the image of its projection into $\mathbb{P}_3$ is nothing but F. This projection is birational (in general two tangent lines to C don't meet) and finite (the tangent lines to C can't all pass through a fixed point). Hence $E=\tilde{F}$ and the projection from E into $\mathbb{P}_3$ is the projection from $\tilde{F}$ onto F.

III. Proof of the Main Result

This section is dedicated to the proof of

Theorem 4. *Given any integer $n\geqq 4$, there exist smooth rational curves C of degree n in $\mathbb{P}_3$ with normal bundle isomorphic to $O_C(2n-1-a)\oplus O_C(2n-1+a)$ if and only if $|a|\leqq n-4$.*

Proof. Let C be a curve as mentioned in the theorem, and let $N_1\subset N_C$ be isomorphic to $O_C(2n-1+a)$, with $a\geqq 0$. Using the canonical projection from $T_{\mathbb{P}_3}|C$ onto N we obtain from N_1 a subbundle V_1 of $T_{\mathbb{P}_3}|C$. This bundle has rank 2 and $\deg(c_1(V))=2n+a+1$. By a theorem of Nakano [S, p. 265] there exists an exact sequence

$$0\to O_C\otimes O_{\mathbb{P}_3}(-1)\to g^*(U)\to V_1\otimes O_{\mathbb{P}_3}(-1)\to 0,$$

where $g\colon C\to\mathbb{P}_3^*$ is the Gauss map of V_1, and U the universal subbundle on $\mathbb{P}_3^*$. Hence $\deg c_1(g^*(U))=-n+1+a$, and $g^*(O_{\mathbb{P}_3^*}(1))=O_C(n-a-1)$. Since C is not a plane curve, g is not constant and we find already $a\leqq n-2$. If $a=n-2$, then $g(C)$ is a line, in other words, all tangents to C meet the same line, which is impossible. For, projecting C from a general point of this line onto a plane would yield a plane curve, the tangents of which pass through a fixed point; this would mean that this projection is a line and C a plane curve. And in the case $a=n-3$ the image $g(C)$ would be either a line or a conic. The first case can be excluded in the same way as before. If $g(C)$ is a conic, then all the tangents to C are contained in tangent planes to a quadratic cone Q. They can't all pass through the vertex v, hence we can project from v (almost all of) these tangents onto a general plane H, and find that all projections are tangent to the conic of intersection $Q\cap H$. This implies that $C\subset Q$. But on Q there are no smooth rational curves of degree $\geqq 4$.

Now let $a \in Z$, with $0 \leqq a \leqq n-4$. We shall construct a smooth rational curve C of degree $n\,(\geqq 4)$ with $a(C)=a$. We distinguish between three cases:

(1) $n-a \equiv 1(3)$. By Proposition 1 there exists a rational curve A of degree $d = \frac{n-a+5}{3}$, spanning $\mathbb{P}_3$, with $l_0(A)=l_1(A)=0$. Let F be the tangent surface of A, let $\tilde{F}$ be its desingularisation and $p\colon \tilde{F} \to F$ the projection. We know by II.4 that $\tilde{F}$ is a $\mathbb{P}_1$-bundle over $\mathbb{P}_1$. Let $\tilde{A}$ be the curve of tangent points on $\tilde{F}$, i.e. the intersection of $\tilde{A}$ with a fibre $\tilde{V}$ consists of all those points, where $p(\tilde{V})$ is tangent to A. Since for a general $\tilde{V}$ the line $p(\tilde{V})$ is tangent to A in exactly one point, $\tilde{A}$ is a section in $\tilde{F}$ (hence $\tilde{A}$ can be identified with the normalisation of A in an obvious way and our notation remains reasonably consistent). Then $p^*(O_F(1)) = [\tilde{A} + \lambda \tilde{V}]$ for some $\lambda \in Z$. Using the fact that $\deg F = 2d-2$ (see II.1) we find

$$(\tilde{A} + \lambda \tilde{V})^2 = 2d-2$$
$$\tilde{A}(\tilde{A} + \lambda \tilde{V}) = d,$$

hence $\tilde{A}^2 = 2$ and $\lambda = d-2$. Therefore $\tilde{F}$ is either isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ or Σ_2*, and $\tilde{A} + k\tilde{V}$ is very ample as soon as $k \geqq 1$. So if we take $k = n-d$, we have that $\tilde{A} + k\tilde{V}$ is very ample and $(\tilde{A} + k\tilde{V})(\tilde{A} + (d-2)\tilde{V}) = n$. In other words, a general curve $\tilde{C}$ in $|\tilde{A} + k\tilde{V}|$ projects onto a curve C of degree n in $\mathbb{P}_3$. This curve is smooth by Proposition 2, for p is finite by Proposition 3. We also know that $p|\tilde{C}\colon \tilde{C} \to C$ is an isomorphism, and we may furthermore assume that F is smooth in a general point of C.

We define a map $a\colon C \to A$ as follows: for any point $c \in C$ we set $a(c)$ equal to the image in A of the point in $\tilde{A}$ lying on the fiber of $\tilde{F}$ through $(p|C)^{-1}(c)$. Less formally, we may describe this map by saying that for general c the tangent line to A in $a(c)$ meets C in c.

Next we define $f\colon C \to \mathbb{P}_3^*$ by setting $f(c)$ equal to the osculating plane of A in $a(C)$. By elementary differential geometry we see that for general c, $f(c)$ is the tangent plane to F at c; hence for any c, $f(c)$ contains the tangent line to C at c. In this way, f defines a 2-sub-bundle V_1 of $T_{\mathbb{P}_3}|C$ which contains T_C as a sub-bundle. Using the Nakano sequence and the fact that $r(A) = 3d-6$ (by II.1), we find that $\deg(c_1(V_1)) = 2n+a+1$. Thus V_1 projects onto a 1-dimensional subbundle of N with degree $2n-1+a$.

(2) $n-a \equiv 0(3)$. In this case we proceed exactly in the same way as in case (1), starting from a curve A of degree $d = \frac{n-a+6}{3}$.

(3) $n-a \equiv 2(3)$. Again we proceed as before, now taking $d = \frac{n-a+7}{3}$.

We shall show in [E−V] that the geometric situation described above is in fact typical; *all* curves in S_i can be obtained by the preceding construction, as divisors on developable surfaces or on cones over curves of suitable class.

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* This is true for every smooth rational curve, as follows from the Nakano sequence

IV. Some Special Results

1. Curves on Quadrics

From many points of view the simplest space curves are those that lie on a quadric surface. Since there are no smooth rational curves of degree $\geqq 4$ on a quadratic cone, we restrict our attention to the smooth quadrics, which are isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The smooth rational curves of degree n on such a surface are the graphs of $(n-1)$-to-1 maps from $\mathbb{P}_1$ to $\mathbb{P}_1$; they lie in the divisor class $n-1$, 1. Since they have infinitely many $(n-1)$-secants, whereas the general rational curve of degree 5 has only one quadrisecant, and the general rational curve of degree $n \geqq 6$ has no $(n-1)$-secants at all, they are very special.

We shall show first that this specialness reflects itself in the restriction of the bundle $T_{\mathbb{P}_3}$ to the curve:

Proposition 5. *A smooth rational space-curve C of degree $n \geqq 3$ is contained in a smooth quadric if and only if $T_{\mathbb{P}_3}|C \simeq O_C(2n-2) \oplus O_C(n+1) \oplus O_C(n+1)$.*

Proof. If C is contained in a smooth quadric Q, then, by the adjunction formula, C is of type $(n-1, 1)$. The pencil of lines on Q, which intersect C in one point, gives rise to 1-dimensional subbundle L of $T_{\mathbb{P}_3}|C$, which has degree $2n-2$ by Nakano's theorem. Let L_p be the 1-subbundle of $T_{\mathbb{P}_3}|C$, which comes from joining the points of C with a fixed point $p \in C$ (in p itself you have to take the tangent line to C). Again by Nakano's theorem we have that $\deg(c_1(L_p)) = n+1$. Given any point $x \in C$ and a 2-dimensional subspace H of $T_{\mathbb{P}_3}(x)$, we can find a bundle L_p, such that $L_p(x) \not\subset H$. Thus a direct sum of bundles of the form L_p will map onto $T_{\mathbb{P}_3}|C$. Therefore, given any decomposition $T_{\mathbb{P}_3}|C \simeq O_C(a_1) \oplus O_C(a_2) \oplus O_C(a_3)$, each of a_1, a_2, and a_3 must be at least $n+1$. On the other hand the existence of L implies that at least one of the a_i's is $2n-2$ or more. Since $a_1 + a_2 + a_3 = 4n$, this leaves us with only the possibility, $T_{\mathbb{P}_3}|C \simeq O_C(2n-2) \oplus O_C(n+1) \oplus O_C(n+1)$, where the first summand is identical with L as soon as $n \geqq 4$.

Conversely, if $T_{\mathbb{P}_3}|C$ contains a direct summand $L \simeq O_C(2n-2)$, then L gives rise to a surface S by way of the Gauss map. The surface S has degree 1 or 2; since C is not contained in a plane, S is a quadric. Further, S can't be a cone if $n \geqq 4$, for there are no smooth rational curves of degree $\geqq 4$ on a quadratic cone. Therefore, if $d \geqq 4$, the curve C is contained in a quadric, such that the Gauss images of the fibres of L form one of the line systems on the quadric. Since every space cubic is contained in a smooth quadric, we have proved the proposition.

On the other hand, the normal bundle of a space-curve on a smooth quadric is the same as that of the general curve:

Proposition 6. *A smooth rational space-curve C of degree $n \geqq 3$ which is contained in a smooth quadric has normal bundle*

$$N_C \simeq O(2n-1) \oplus O(2n-1).$$

This result was first noted by Harris [Ha]. In [H], Hulek analyses the normal bundles of all smooth curves on quadrics in $\mathbb{P}_3$, and in particular obtains a new proof of our result.

We shall give a geometric proof, and then an algebraic proof, which makes the result a consequence of local duality.

Proof of Proposition 6. Since every smooth rational curve of degree 4 has a balanced normal bundle (Theorem 4), we can assume that $n \geqq 5$.

Again, let L be the 1-subbundle of $T_{\mathbb{P}_3}|C$, given by the pencil of lines on the quadric Q, which intersect C in one point. Since such a line is never tangent to C, we obtain, by projecting L into N, a 1-subbundle L' of N with $\deg(c_1(L')) = 2n-2$. Consequently, either N is balanced, or $N \simeq \mathcal{O}_C(2n) \oplus \mathcal{O}_C(2n-2)$. All we have to do is to exclude the second possibility.

Suppose that N has a 1-subbundle $M \simeq \mathcal{O}_C(2n)$ (such a bundle is necessarily unique). Then $N = L' \oplus M$, and the inverse image of M in $T_{\mathbb{P}_3}|C$ is a 2-bundle V with $\deg(c_1(V)) = 2n+2$. It follows from Proposition 5 that $V \simeq \mathcal{O}_C(n+1) \oplus \mathcal{O}_C(n+1)$. So $T_{\mathbb{P}_3}|C$ has a family $\{L_t\}_{t \in P_1}$ of 1-subbundles isomorphic to $\mathcal{O}_C(n+1)$. By way of its Gauss map each of these surfaces yields an irreducible ruled surface W_t. The degree of W_t is either $n-1$ or at most $\frac{1}{2}(n-1)$. In the second case W_t has to be identical with Q, for if an irreducible surface of degree $\leqq \frac{1}{2}(n-1)$ and an irreducible quadric have a curve of degree n in common, they must coincide. But W_t can't be identical with Q, for in a general point $c \in C$ the line $g_{L_t}(c)$ is not contained in Q. Therefore all surfaces W_t are of degree $n-1$. They intersect Q in a divisor of type $(n-1, n-1)$, the support of which consists of at most two irreducible curves [a line $g_{L_t}(c)$, which is not contained in Q, intersects Q in c and one other point]. Hence $Q \cap W_t = C \cup (n-2) R_t$, where R_t is a line on Q. In this way we obtain a $1-1$-correspondence between the subbundles L_t (or the surfaces W_t) and the pencil of lines on Q, for which $R_t C = n-1$. We take t general, such that in particular R_t and C meet transversally in $n-1$ points. The Gauss map g_{L_t} gives a map $f\colon C \to R_t$. This map can't be constant; otherwise W_t would be a cone with vertex $v \in R_t$, and there would always be a point $x \in C \cap R_t$, $x \neq v$, such that $g_{L_t}(x) = R_t$, which is impossible (L' and M have no fibre in common). So f is a map of degree $n-2$. For every point $y \in C \cap R_t$ we have $f(y) = y$, otherwise $g_{L_t}(y)$ would again be R_t. But we claim more, namely that $g_{L_t}(y)$ is tangent to C at y. This can be seen in the following way. For a general point $x \in C$ the plane through x and R_t is certainly not the plane $V(x)$, hence $L_t(x)$ is the intersection of these two planes. Therefore $g_{L_t}(y)$ will be the intersection of the limit positions of these two planes if x approaches y, provided of course that these limit positions are different. The limit position of the first plane is the plane through R_t and the tangent line to C at y. The second plane contains this tangent line, but certainly not R_t. Hence $L_t(y) = T_C(y)$.

Now take the other coordinate z on Q as a coordinate on both C and R_t. Then f can be seen as a map from $P_1(z)$ onto itself of degree $n-2$. We may assume that $f(\infty) \neq \infty$. An easy calculation shows that $f'(y) = 0$ for every $y \in C \cap R_t$. Hence $f \equiv \dfrac{p_1(z)}{p_2(z)}$, where $p_1(z)$ and $p_2(z)$ are (inhomogeneous) polynomials of degree $n-2$ in z, without common roots. Let z_0 be a value of z, corresponding to a point of $R_t \cap C$. We have $p_1(z_0)-z_0 p_2(z_0)=0$ and also $p'_1(z_0)p_2(z_0)-p'_2(z_0)p_1(z_0)=0$, hence $p'_1(z_0)-z_0 p'_2(z_0)=0$. But this means that this last equation, which is of degree at most $n-2$ would have $n-1$ different roots, since by assumption C and R_t have $n-1$ different points in common. This contradiction shows that our assumption, namely the existence of a 1-subbundle L of N with $L \simeq O_C(2n)$ is impossible. Hence N_C is always balanced if $C \subset Q$.

The idea behind this proof can be used to study the normal bundle of any smooth space curve. Let C be such a curve and $L \subset N_C$ a 1-dimensional subbundle. Let V be the inverse image of L in $T_{\mathbb{P}_3}|C$. If we take a general line $R \subset \mathbb{P}_3$ [such that R is not contained in any plane $g_V(x)$, $x \in C$], then V gives rise to a map f: $C \to R$ by taking for $f(x)$ the point $g_V(x) \in R$. The degree of f can be expressed in terms of $\deg c_1(L)$. This map has the following property: if $y \in C$, such that the tangent line to C at y intersects R in z, then $f(y)=z$. Moreover, the limiting position of the line through x and $f(x)$ if x approaches y is the tangent line to C at y. Thus the existence of special 1-subbundles of N leads to the existence of special correspondences. A certain converse also holds, though it should be stated with care. We hope to return to this point at another time.

Sketch of an Algebraic Proof of Proposition 6. Suppose C is contained in the quadric $X_0X_2-X_1X_3=0$. Realising C as the graph of an $(n-1)$-to-1 map, we may write it parametrically as

$$\mathbb{P}_1 \ni (s,t) \to (p_0, p_1, p_2, p_3) \in \mathbb{P}_3,$$

where each p_i is a form of degree n in 2 variables, and $p_0=sp$, $p_1=sq$, $p_2=tp$, $p_3=tq$ for some relatively prime forms p and q.

In this situation the normal bundle of C is determined by the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & O_C \arrow[r, "(n-1)"] \arrow[d, "{(s,t)}"'] & O_C \arrow[d, "{(p_0,p_1,p_2,p_3)}"] & & \\
 & O_C^2(1) \arrow[r, "\mathcal{J}"] \arrow[d, "\binom{t}{-s}"'] & O_C^4(n) \arrow[d, "\kappa"] & & \\
0 \arrow[r] & T_C = O(2) \arrow[r, "\phi"] \arrow[d] & T_{\mathbb{P}_3}|C \arrow[r] \arrow[d] & N_C \arrow[r] & 0, \\
 & 0 & 0 & &
\end{tikzcd}$$

where

$$\mathscr{J} = \frac{\partial(p_0, p_1, p_2, p_3)}{\partial(s,t)}$$

is the Jacobian matrix.

Applying the functor $F = \sum_{v} H^0(C, O_C(v))$, and writing $S(k)$ for the free $\mathbb{C}[s,t]$-module whose generator has degree $-k$, we may turn the computation of N into a problem on graded modules. One sees easily that $F(\kappa)$ is the map

$$S(-n-1) \oplus S(-n-1) \oplus S(-2n:2) \to S^4(-n)$$

given by the matrix

$$\begin{pmatrix} t & 0 & \frac{\partial q}{\partial s} \\ -s & 0 & \frac{\partial q}{\partial t} \\ 0 & t & \frac{-\partial p}{\partial s} \\ 0 & -s & \frac{-\partial p}{\partial t} \end{pmatrix}$$

and a simple calculation then shows that

$$F(\phi): S(-n-1) \oplus S(-n-1) \oplus S(-2n+2) \to S(-2)$$

is given by the matrix

$$\left(p, q, \frac{1}{t}\left(p\frac{\partial q}{\partial s} - q\frac{\partial p}{\partial s}\right) + J\right),$$

where $J = \det \begin{pmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{pmatrix}$, and the expression $\left(p\frac{\partial q}{\partial s} - q\frac{\partial p}{\partial s}\right)$ is formally divisible by t.

Changing coordinates we may assume $p = tr$ for some form r of degree $n-2$, and then the above matrix can be rewritten $(p, q, J_0 + J)$, where

$$J_0 = \det \begin{pmatrix} \frac{\partial q}{\partial s} & q \\ \frac{\partial r}{\partial s} & r \end{pmatrix}.$$

We now note that $J = (1-n)J_0$; since both J and J_0 are multilinear in r and q, it suffices to check this for monomials, where it is evident. Thus the above matrix has the form

$$F(\phi)=\left(p, q, \frac{(2-n)}{(1-n)} J\right).$$

By the local duality theorem (see for example [G-H, 5.1]) J generates the unique minimal ideal of $\mathbb{C}[s,t]/(p,q)$ and it follows that $\ker F(\phi)=S(-2n+1) \oplus S(-2n+1)$.

Translating this into the language of bundles, we find

$$N=O_C(2n-1)\oplus O_C(2n-1)$$

as required.

2. *Realising the Splitting of the Normal Bundle*

Let C be a smooth rational curve of degree 3. Fixing $p\in C$, we can construct a 2-bundle V_p on C as follows. If $x\in C$, $x\neq p$, then we take for $V_p(x)$ the 2-dimensional linear subspace of $T_{\mathbb{P}_3}|C(x)$, determined by the well-defined plane through p and the tangent line to C at x; if $x=p$ we use instead the osculation plane to C at p. The bundle V_p gives a 1-subbundle $L_p\subset N=N_{C/\mathbb{P}_3}$, and $N\cong L_p \oplus L_q$ if $p\neq q$. In fact, $N\cong O_C(5)\oplus O_C(5)$ and the pencil of 1-subbundles of N, which are isomorphic to $O_C(5)$ is exactly the pencil $\{L_p\}$, $p\in C$.

All this is easy to verify.

We can identify the projective bundle $P=P_C$ of N with the bundle attaching to each point $x\in C$ the projective line of planes, passing through the tangent line of C at x. Then a splitting of N consists of two sections in this bundle, which don't meet. In other words, a splitting consists of two fields of planes H_1, H_2 along C, such that both $H_1(x)$ and $H_2(x)$ always contain the tangent line to C at x, but $H_1(x)\neq H_2(x)$ for all $x\in C$. In this light the preceding construction for a cubic can be interpreted in the following way. We start by taking a fixed point $p\in C$. If $x\neq p$ we take for $H_p(x)$ the tangent plane to the quadratic cone over C with vertex p; for $x=p$ we take $H_p(p)=\lim_{x\to p} H_p(x)$. If we carry out this construction for two different points of C, we obtain a splitting of N.

Given any 1-subbundle of N, in other words a section s in P, then for k large enough there always exist surfaces S of degree k (containing C and singular in a finite number of points on C) such that the field of tangent planes to S along C (plus limiting positions) yields the section s. This is a consequence of the fact that the natural homomorphism

$$\Gamma(C,\mathscr{I}(k))\to\Gamma(C,\mathscr{I}(R)/\mathscr{I}^2(k))=(C,N^*(k))$$

is surjective for $k\geq k_0$ (here $\mathscr{I}$ denotes the ideal sheaf of C).

So given a smooth rational curve C we can always realise the splitting of N_C using surfaces of sufficiently high degree. Is it possible to do so in a simple way, using surfaces of a degree as low as possible, as was done before with quadratic cones for a cubic? We don't know a general answer to this (admittedly not very precise) question, but we know an attractive answer in several special cases. As an example we treat the case of quartics. By the Main

Theorem the normal bundle is always isomorphic to $O_C(7)\oplus O_C(7)$ for such a curve C.

Let C_{norm} be a smooth rational quartic in $\mathbb{P}_4$. We always can find a point $c\in\mathbb{P}_4 (c\notin C_{\mathrm{norm}})$ and linear embedding of $\mathbb{P}_3$ in $\mathbb{P}_4$, such that the projection f from c onto $\mathbb{P}_3$ maps C_{norm} isomorphically onto C. Now C is contained in a smooth quadric as a curve of type $(3,1)$. The planes through c which meet C_{norm} in three points correspond by f to the lines on Q which meet C in three points. Let V be such a plane, meeting C_{norm} in three different points p_1, p_2, and p_3. Since no three of the four points p_1, p_2, p_3 and c are collinear, there exists a smooth conic $K\subset V$ passing through p_1, p_2, and p_3, such that its tangents in p_1 and p_2 meet in c.

We consider the uniquely determined linear correspondence between the points of K and those of C_{norm}, for which p_1, p_2, and p_3 correspond to themselves. The union of the lines joining corresponding points (plus limit positions) is a scroll, i.e. a ruled surface of degree 3, isomorphic to $\mathbb{P}_2$ with one point blown up; one checks this most easily by showing that the degree of the surface is 3 and checking the possibilities. The lines on R determine a 1-subbundle of $T_{\mathbb{P}_4}|C_{\mathrm{norm}}$, which is isomorphic to $O_C(5)$ by Nakano's theorem. Projecting from c we obtain a 1-subbundle of $T_{\mathbb{P}_3}|C$, which is tangent to C in at least two points, namely $f(p_1)$ and $f(p_2)$. Further projection into N_C gives a 1-subbundle N_1, of degree 7 or more. Since N_C does not contain any 1-subbundles of degree 8 or more, we find that $N_1\cong O_C(7)$. Its inverse image in $T_{\mathbb{P}_3}|C$ in a general point $x\in C$ can be obtained as f [tangent plane to R at $f^{-1}(x)$]. On the other hand, $f(R)$ is the unique Steiner surface in $\mathbb{P}_3$, which has $f(V)$ as its double line and $f(p_1)$, $f(p_2)$ as its pinch points [G-H, p. 629]. Thus the tangent planes along C of two of these Steiner surfaces give a splitting of N_C. To obtain all 1-subbundles of N_C, isomorphic to $O_C(7)$ one has to consider some limit cases.

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References

[B] Brieskorn, E.: Über holomorphe $\mathbb{P}_n$-Bündel über $\mathbb{P}_1$. Math. Ann. **157**, 343–357 (1965)

[E-V] Eisenbud, D., Van de Ven, A.: On the family of smooth rational space curves with given degree and normal bundle.

[G-H] Griffiths, P.A., Harris, J.: Principles of algebraic geometry. New York: Wiley 1978

[G-S] Ghione, F., Sacchiero, G.: Normal bundles of rational curves in $\mathbb{P}_3$ (preprint)

[Ha] Harris, J.: A note on the normal bundles of space curves (unpublished)

[Hu] Hulek, K.: The normal bundle of a curve on a quadric (preprint, May 1981)

[K] Kodaira, K.: A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. Ann. Math. **75**, 146–162 (1962)

[P] Piene, R.: Numerical characters of a curve in projective n-space. In: Real and complex singularities. Oslo 1976. Groningen: Sijthoff en Noordhoff

[S] Simonis, J.: A class of indecomparable algebraic vector bundles. Math. Ann. **192**, 262–278 (1971)

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