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Young Diagrams and Determinantal Varieties

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Introduction

Let F be a commutative ring and fix integers $m \geqq n \geqq 1$. Set

$$R = F[\{X_{ij}\}_{1 \leqq i \leqq n,\, 1 \leqq j \leqq m}],$$

the polynomial ring on mn variables X_{ij}. If we regard $X = (X_{ij})$ as an $n \times m$ matrix ("the generic $n \times m$ natrix over F") we can define an action of the group $G = GL(n, F) \times GL(m, F)$ on R by the formula $(A, B)\, X_{ij} = X'_{ij}$, where $A \in GL(n, F)$, $B \in GL(m, F)$, and X'_{ij} is the ij^{th} entry of the matrix $A^{-1} XB$. In this paper we study the arithmetic of the ideals of R that are invariant under the action of G.

Our program is motivated by geometric and invariant-theoretic considerations. The ring R is the coordinate ring of the space $M_{n,m} = \mathrm{Hom}_F(F^m, F^n)$, on which G acts by $(A, B)\, \phi = A\, \phi\, B^{-1}$ for $\phi \in M_{n,m}$, and the induced action on R is the one described above. Supposing, say, that F is a field, the action of G on $M_{n,m}$ is easy to describe: the orbits are the sets of maps with given rank, and the closure V_k of the orbit of maps of rank k is the set of maps of rank $\leqq k$. However, the action of G on R is more interesting. For example, it is a well-known but non-trivial theorem, closely connected to invariant theory, that the ideal of

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functions vanishing on V_k is I_{k+1}, the ideal generated by the $k+1$-order minors (=subdeterminants) of X, and that the coordinate ring R/I_{k+1} of V_k is normal and Cohen-Macaulay ([Eagon-Hochster], [Kempf], [Musili]). Its singularities occur naturally in classical problems [Kempf]. Although information about the ideals I_{k+1} is relatively complete, there are many problems concerning, for example, the ideals $I_{k+1}^{(p)}$ of functions vanishing to order $\geqq p$ on V_k [Hochster 1]. More generally, it is natural to ask about the ideals of functions vanishing to prescribed order on each of the varieties V_k, as well as, for instance, about products of the ideals I_k. These ideals are clearly all G-invariant, and will be described in this paper.

Our method is to exploit the representation theory of G, often via the beautiful characteristic-free reformulation in terms of standard basis and straightening formula in [Doubilet-Rota-Stein]. (See the end of this introduction for an historical sketch.) The idea is, roughly speaking, that one may find an F-free basis of R by taking certain "standard monomials" in the minors of X, using the larger minors "wherever possible". (The usual basis for the polynomial ring R, by contrast, consists of all the monomials in the 1×1 minors - the elements X_{ij}.) The straightening formula, of which we give a new proof (of an improved version due to W. Hesselink and, independently, J. Stein) in Sect. 2, expresses the non-standard monomials in terms of the standard ones in a nice way.

The standard basis of R is intimately connected with the action of G. In the case where F is a field of characteristic 0 we supplement the straightening formula techniques by using the decomposition of R into G-irreducible sub-modules, and the fact that each standard basis element is associated to one of these. Our results in this case are correspondingly more precise.

We now describe our results, supposing for simplicity that F is a field of characteristic 0, although we will do many things for arbitrary F.

Let $\mathbb{N}$ be the set of natural numbers. A (Young) *diagram* is a finite subset $\sigma \subset \mathbb{N} \times \mathbb{N}$ such that if $(i,j) \in \sigma$ and $i' \leqq i$, $j' \leqq j$, then $(i',j') \in \sigma$. Diagrams correspond to certain irreducible representations L_σ and ${}_\sigma L$ of $GL(n,F)$ and $GL(m,F)$, respectively. As a G-module, the polynomial ring R decomposes as

$$R = \Sigma M_\sigma, \tag{*}$$

where $M_\sigma \cong L_\sigma \otimes_F {}_\sigma L$, and the sum ranges over all diagrams σ such that $(i,j) \in \sigma$ implies $j \leqq n$ (the diagrams "with $\leqq n$ columns"). This (old result) is proved in Sect. 3, where the modules L_σ are also described explicitly.

We may now describe a G-invariant ideal of R by the set of M_σ it contains. For any set J of diagrams of at most n columns, we set $I(J) = \sum_{\sigma \in J} M_\sigma$. Our first main result is that $I(J)$ is an ideal if and only if J is an ideal in the set of diagrams (a *D-ideal*) in the sense that $\sigma \in J$ and $\tau \supseteq \sigma$ imply $\tau \in J$ (4.1 and 5.1).

There is a close relation between the arithmetic of diagrams and that of ideals. We describe the primeness, primaryness, primary decomposition and integral closure of $I(J)$ are described in terms of J (5.2, 5.3, 5.4 and following remarks, 8.2).

Among the G-invariant ideals, the most significant are the ones associated to determinantal ideals. For any diagram σ we write $\sigma_i = \max\{j \mid (i,j) \in \sigma\}$ for the *length of the* i^{th} *row of* σ, and set $D_\sigma = I_{\sigma_1} I_{\sigma_2} \ldots$. It turns out that the D-ideal corresponding to D_σ is the ideal $\{\tau \mid \tau \geqq \sigma\}$ (Theorem 6.1), where $\geqq$ denotes the Snapper (partial) order on diagrams (defined in Sect. 1).

One question that has inspired past work is the question of the primary decomposition of the powers of the prime ideals I_k.

Partial results have been obtained in [Hochster], [Achilles-Schenzel-Vogel], and [Trung]. We complete the picture by identifying the D-ideals corresponding to the symbolic powers $I_k^{(p)}$ of I_k (which may be thought of as the I_k-primary-components of I_k^p, or again as the ideals of functions vanishing to order $\geqq p$ along the determinantal variety V_{k-1}), and showing that

$$I_k^p = I_k^{(p)} \cap I_{k-1}^{(2p)} \cap \ldots \cap I_1^{(kp)},$$

with irredundancy if the terms involving I_s are dropped for $s < n - p(n-k)$ (Corollary 7.3).

Our most important "characteristic-free" innovation is the introduction (for arbitrary F) of a filtration of R by ideals A_σ generated by certain products of minors of the matrix X (definition at the end of Sect. 2). If F is a field of characteristic 0, then $A_\sigma = D_\sigma$, but in general A_σ is a little larger than D_σ (6.1). We use this filtration as a characteristic free version of *); if A'_σ denotes $\sum_{\tau > \sigma} A_\tau$, then $A_\sigma / A'_\sigma \cong L_\sigma \otimes_F {}_\sigma L$ for a suitable characteristic-free version of L_σ and ${}_\sigma L$.

We will now sketch some of the ideas behind the straightening formula and the decomposition of R.

The study of the action of G on R was one of the main concerns of classical invariant theory. The 19th century work can be found in A. Capelli's book [Capelli] (see also [Deruyts]). The main results can be summarized as follows:

The Lie algebra $\mathfrak{g}$ of $GL(m)$ acts on R in a natural way as an algebra of differential operators. In classical terminology, with characteristic $F = 0$, this is the algebra generated by the polarization operators. The elements of $\bigoplus L_\sigma \subset R$ are called the *primary* covariants of $GL(m)$ in R. Any element may be obtained from elements in this submodule by applying the operators in $\mathfrak{g}$; this is part of the Capelli-Deruyts expansion, a generalization of the Clebsch-Gordan expansion.

The matter became clearer with the theory of group representations, in particular with Schur's thesis. Schur proved (always with char $F = 0$) that the L_σ, for various σ, yield the distinct irreducible polynomial representations of $GL(n)$. He showed that for each $h \geqq 0$, $GL(n)$ spans a semisimple algebra of operators on the space R_h of forms of degree h. Of course similar statements hold for $GL(m)$ and ${}_\sigma L$. Each of the algebras generated by $GL(n)$ and $GL(m)$ is the centralizer of the other, so we get, in this case, $R_h \cong \sum_\sigma L_\sigma \otimes_F {}_\sigma L$ canonically, where σ runs over the diagrams of $\leqq n$ columns such that $\Sigma \sigma_i = h$. Thus we get the formula (*).

This formula can also be proved by using characters; the characters of the L_σ were computed by Schur, and are symmetric functions in the eigenvalues that were introduced by Jacobi. The decomposition (*) is equivalent to an identity on symmetric functions which is due to Cauchy.

In Sect. 3 we give a definition of L_σ in the case $F=Z$. This is just one possible choice for an "integral form" of the irreducible $GL(n, Q)$-module $L_\sigma \otimes Q$. One of the reasons that this choice is an interesting one is that it is connected with line bundles on the flag variety [Seshadri]: Let $\mathscr{F}$ be the variety of complete flags in F^n; we write λ_k for the k^{th} exterior power of the tautological k-plane bundle on $\mathscr{F}$. To each diagram σ we associate the line bundle $\lambda_\sigma = \lambda_{\sigma_1} \otimes \lambda_{\sigma_2} \otimes \ldots$. There is a canonical isomorphism of L_σ with $H^0(\mathscr{F}, \lambda_\sigma^*)$ (a special case of the Borel-Weil theorem), valid over any ring F; it comes from the invariant theory of the unipotent radical of a Borel subgroup in $GL(n)$, and could be deduced, for example, from the results of Sect. 3.

An important step in the theory was the discovery by A. Young [Young] of special bases for the representations of the symmetric and general linear groups, using standard tableaux (see Sect. 1, below). His theory, recast in geometric language in [Hodge], was used in [Igusa] for proving the first main theorem of invariant theory of the special linear group.

Doubilet-Rota-Stein discovered a useful generalization of Young's theory (the straightening formula of [Doubilet-Rota-Stein]), which is the basis of our work. Its power was shown in [De Concini-Procesi] for classical invariant theory, and was generalized in the spirit of Schubert calculus, as in Hodge's treatment in [Lakshmibai-Musili-Seshadri], and in [De Concini]. Another description of the L_σ was given in [Towber].

We are very grateful to W. Hesselink who uncovered and helped repair several gaps and many minor obscurities in an earlier version of this work. In addition to the improved version of the Straightening formula which he communicated to us, the statement and proof of Proposition 1.4 are due to him.

1. Combinatorics of Diagrams and Tableaux

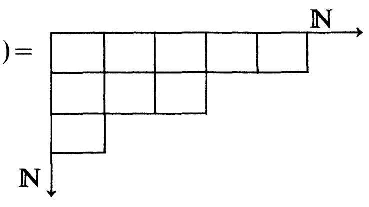
A. Diagrams

A diagram σ (defined in the introduction) is determined by its sequence of *row lengths* $\sigma_i = \max\{j \mid (i,j) \in \sigma\}$ and also by its sequence of *column lengths* $\check{\sigma}_j = \max\{i \mid (i,j) \in \sigma\}$. (We take the max of the empty set to be 0.) These form decreasing sequences. We will often identify σ with the sequence $(\sigma_1, \sigma_2, \ldots)$. The *dual* diagram, obtained by interchanging the factors in $\mathbb{N} \times \mathbb{N}$, is called $\check{\sigma}$; thus $\check{\sigma}_j$ is also the length of the j^{th} row of $\check{\sigma}$.

By the *number of columns of* σ we mean the smallest $n \in \mathbb{N}$ such that $\sigma_i \leqq n$ for all i, or equivalently, such that $\check{\sigma}_{n+1} = 0$; the *number of rows of* σ is defined dually.

We think of σ as a sequence of rows of "boxes" of lengths $\sigma_1, \ldots$ (the "Ferrers diagram"). For example

$\sigma = (5, 3, 1) =$



Containment among subsets of $\mathbb{N}\times\mathbb{N}$ provides one partial order, written $\subseteq$, on the set of diagrams. Another, the "Snapper" order, written $\leqq$, is defined as follows:

To a diagram σ we associate the two sequences (either of which determines σ)

$$\beta_k(\sigma)=\sum_{i=1}^{k}\sigma_i$$

and

$$\gamma_k(\sigma)=\sum_{j=k}^{\infty}\check{\sigma}_j$$

(note that almost all the $\check{\sigma}_j$ are 0!).

If σ and τ are both diagrams, then we write

$$\sigma\geqq_\beta\tau \text{ iff } \beta_j(\sigma)\geqq\beta_j(\tau) \quad \text{for all } j,$$
$$\sigma\geqq_\gamma\tau \text{ iff } \gamma_j(\sigma)\geqq\gamma_j(\tau) \quad \text{for all } j.$$

As is well-known, the partial orders $\geqq_\beta$ and $\geqq_\gamma$ are the same. After the proof of Proposition 1.1 we will write $\geqq$ for $\geqq_\beta$ or $\geqq_\gamma$.

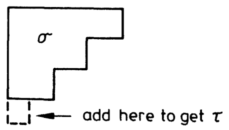
Proposition 1.1. $\sigma\geqq_\beta\tau$ iff $\sigma\geqq_\gamma\tau$.

Before proving this, we need another result. We will say that $\tau>_\beta\delta$ are adjacent if there is no diagram τ' with $\tau>_\beta\tau'>_\beta\sigma$.

The following analysis of adjacency sharpens and generalizes, for the case of integral vectors (=diagrams) Lemma 2, p. 47, of [Hardy-Littlewood-Polya]. (Theorems 45, 46, 47, 74, 75 of that work will also interest the reader who wishes to know more about $\leqq$, as will section 1.1 of [Gerstenhaber]. A proof of Proposition 1.1 can easily be given from Gerstenhaber's Theorem 3.)

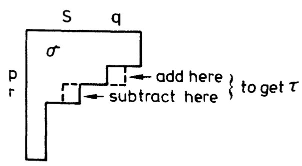
Proposition 1.2. *If σ and τ are diagrams, then $\tau>_\beta\sigma$ are adjacent iff either*

i) $\sigma_p=0$ *and* $\tau=\sigma\cup\{(p,1)\}$*:*

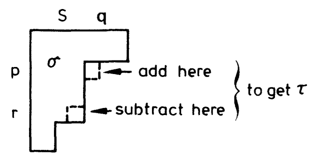


or $\tau=\sigma\cup\{p,q)\}-\{(r,s)\}$, *with* $(r,s)\in\sigma$ *but* $(p,q)\notin\sigma$, *and either*

ii) $r=p+1$*:*



or iii) $q=s+1$:



Remark. We are of course regarding σ, τ as subsets of $\mathbb{N}^2$. If σ is a diagram and $(p, q)\notin\sigma$, then the condition that $\sigma\cup\{(p, q)\}$ be a diagram is that $\sigma_{p-1}>\sigma_p$ and $q=\sigma_p+1$. If $(r, s)\in\sigma$, and $r=p+1$ as in ii, then the set $\sigma\cup\{(p, q)\}-\{(r, s)\}$ is a diagram iff in addition $\sigma_{p+1}>\sigma_{p+2}$ and $s=\sigma_r$. If on the other hand $q=s+1$, then the additional condition is $r>p$, $\sigma_p=\sigma_{p+1}=\ldots=\sigma_r=s>\sigma_{r+1}$.

Proof of Proposition 1.2. We first check that if $\sigma<_\beta\tau$ satisfy i), ii), or iii), then they are adjacent. In cases i) and ii), $\beta_i(\sigma)$ differs from $\beta_i(\tau)$ for only one interesting index i, and for this i, $\beta_i(\tau)=\beta_i(\sigma)+1$, proving adjacency.

If σ and τ are related as in iii), then

$$\beta_i(\tau)=\begin{cases}\beta_i(\sigma) & \text{for } 1\leqq i<p \text{ and for } r\leqq i\\ \beta_i(\sigma)+1 & \text{for } p\leqq i<r.\end{cases}$$

Further, since $q=s+1$, we have $\sigma_i=\sigma_{i-1}$ for $p<i\leqq r$.

If now for some diagram τ' we had $\sigma<_\beta\tau'<_\beta\tau$, then for some u with $p<u<r$ we would have

$$\beta_i(\tau')=\beta_i(\sigma) \quad \text{for } i<u,$$

$$\beta_u(\tau')=\beta_u(\sigma)+1.$$

In particular this implies

$$\tau'_{u-1}=\sigma_{u-1}=\sigma_u=\tau'_u-1<\tau'_u.$$

On the other hand, since τ' is a diagram, $\tau'_{u-1}\geqq\tau'_u$, a contradiction; thus σ and τ are adjacent as claimed.

We next suppose $\tau>_\beta\sigma$ are adjacent. Let p be the first integer for which $\tau_p\neq\sigma_p$, so that $\sigma_{p-1}=\tau_{p-1}\geqq\tau_p>\sigma_p$.

If $\sigma_{p+1}=0$, then either $\sigma_p=0$, in which case $\sigma<\tau'=\sigma\cup\{(p, 1)\}\leqq\tau$, or $\sigma_p>0$, in which case $\sigma<\tau'=\sigma\cup\{(p+1, 1)\}\leqq\tau$. In either case we have $\tau=\tau'$, as in *i*.

If $\sigma_{p+1}>\sigma_{p+2}$, then with $r=p+1$, $s=\sigma_r$, $q=\sigma_p+1$, the set $\tau'=\sigma\cup\{(p, q)\}-\{(r, s)\}$ is a diagram such that $\beta_i(\sigma')$ differs from $\beta_i(\sigma)$ only for $i=p$, and $\beta_p(\tau')=\beta_p(\sigma)+1$. Thus $\tau\geqq_\beta\tau'>_\beta\sigma$ so $\tau=\tau'$ and we are in case ii).

Finally, if $0\neq\sigma_{p+1}=\sigma_{p+2}$, we let p' be the first index for which $\sigma_{p'}=\sigma_{p+1}$, so that $p'=p$ or $p+1$ (it will turn out that $p'=p$). Let $r\geqq p+2$ be the last integer such that $\sigma_r=\sigma_{p+1}$. The diagram

$$\tau'=\sigma\cup\{(p', \sigma_{p'}+1)\}-\{(r, \sigma_r)\}$$

satisfies iii), so $\tau'>_\beta\sigma$ is adjacent.

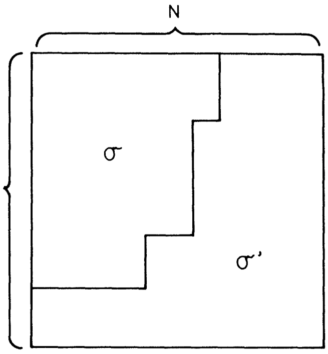
It remains to show $\tau\geqq_\beta\tau'$, and for this it suffices to prove $\beta_i(\tau)>\beta_i(\sigma)$ for $p'<i<r$. Suppose on the contrary that for some first integer u with $p'<u<r$ we

had $\beta_u(\tau)=\beta_u(\sigma)$. Then $\tau_{u+1}\leqq\tau_u<\sigma_u=\sigma_{u+1}$, so $\beta_{u+1}(\tau)<\beta_u(\sigma)$, contradicting $\tau>_\beta\sigma$. This gives $\tau\geqq_\beta\tau'$, whence $\tau=\tau'$ (and $p'=p$); we are in the case iii).

Proof of Proposition 1.1. Let N be a number $\geqq$ the numbers of rows and columns of σ and τ. From any diagram σ of $\leqq N$ rows and columns we may form a new diagram

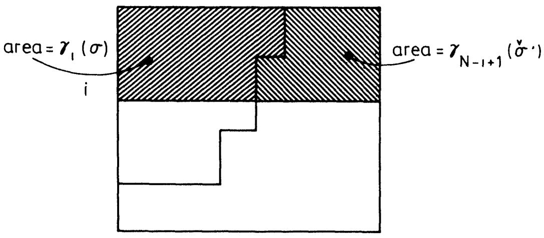
$$\sigma'=(N-\sigma_N, N-\sigma_{N-1}, \dots, N-\sigma_1).$$

It is easy to check that the operations ′, and ˇ commute. Pictorially we have



where σ' appears here with its rows in inverse order. It is easy to check that $\beta_i(\sigma)=N\,i-\gamma_{N-i+1}(\check{\sigma}')$, and similarly that $\gamma_i(\sigma)=N(N-i+1)-\beta_{N-i+1}(\check{\sigma}')$.

Proof that $\beta_i(\sigma)=N\,i-\gamma_{N-i+1}(\sigma')$.



Thus

$$\tau\geqq_\beta\sigma\Leftrightarrow\check{\tau}'\leqq_\gamma\check{\sigma}',$$

and

$$\tau\geqq_\gamma\sigma\Leftrightarrow\check{\tau}'\leqq_\beta\check{\sigma}'.$$

Hence it suffices to show that $\tau\geqq_\beta\sigma$ implies $\tau\geqq_\gamma\sigma$. But if $\tau>_\beta\sigma$, then there is a finite unrefinable chain between τ and β. Thus we need only check that $\tau\geqq_\gamma\sigma$ in case $\tau>_\beta\sigma$ are adjacent; and this is trivial from Proposition 1.2.

We now define a multiplication on diagrams as follows: if $\sigma=(\sigma_1,\dots,\sigma_h)$ and $\tau=(\tau_1,\dots,\tau_k)$ are diagrams with respectively h and k nonzero rows, then $\sigma\tau$ is the diagram with $h+k$ nonzero rows whose set of nonzero row lengths is

$$\{\sigma_1, \ldots, \sigma_h, \ \tau_1, \ldots, \tau_k\}.$$

We next show that the multiplication of diagrams preserves the order $\leqq$, as well as the usual lexicographic order on diagrams.

Proposition 1.3. *If σ, τ, and τ' are diagrams then*
1) $\sigma\tau \geqq \sigma\tau'$ iff $\tau \geqq \tau'$ *and*
2) *If τ is later than τ' in the lexicographic order, then $\sigma\tau$ is lexicographically later than $\sigma\tau'$.*

Proof. 1) We have $\gamma_i(\sigma\tau) = \gamma_i(\sigma) + \gamma_i(\tau)$ for each i, and this trivially implies the result.

2) To say that τ is lexicographically later than τ' means that if i is the smallest number with $\tau_i \neq \tau'_i$, then $\tau_i > \tau'_i$. If the first row of $\sigma\tau$ is σ_1, that is, $\sigma_1 \geqq \tau_1$ then also $\sigma_1 \geqq \tau'_1$, and we reduce to showing that $\sigma'\tau$ is later than $\sigma'\tau'$, where $\sigma' = (\sigma_2, \ldots)$ is a diagram with fewer rows than σ; we are thus done by induction. Thus we may assume $\tau_1 > \sigma_1$.

If $\sigma_1 \geqq \tau'_1$, then the first row of $\sigma\tau'$ has length $\sigma_1 < \tau_1$, and we are done. If, on the other hand $\sigma_1 < \tau'_1$, $\sigma\tau'$ begins with τ'_1. If $\tau_1 > \tau'_1$ we are done; otherwise $\tau_1 = \tau'_1$ and we may reduce to the case of diagrams τ and τ' with fewer rows by dropping the first row of each.

The next result exhibits the minimal data necessary to conclude that a diagram is $\geqq$ a given diagram.

Proposition 1.4. *Let σ be a diagram of at most n columns. There is a unique minimal subset $L \subset \{1, 2, \ldots, n\}$ such that, for all diagrams τ of at most n columns, $\tau \geqq \sigma$ iff*

$$\gamma_k(\tau) \geqq \gamma_k(\sigma) \qquad \textit{for all } k \in L.$$

L is the set of those l such that $\check{\sigma}_l \neq 0$ and $\gamma_{l+1}(\sigma) \leqq (n-l)(\check{\sigma}_l - 1)$.

Remark. For $l \neq n$, the condition

$$\gamma_{l+1}(\sigma) \leqq (n-l)(\check{\sigma}_l - 1) \quad \text{implies } \check{\sigma}_l \neq 0.$$

Proof. Note that for any diagram τ of at most n columns, and any integer l, we have

$$\gamma_{l+1}(\tau) = \check{\tau}_{l+1} + \ldots + \check{\tau}_n \leqq (n-l)\, \check{\tau}_{l+1}.$$

If l is such that $\gamma_l(\tau) < \gamma_1(\sigma)$ and $\gamma_{l+1}(\tau) \geqq \gamma_{l+1}(\sigma)$ then since $\gamma_l(\tau) = \gamma_{l+1}(\tau) + \check{\tau}_l$, and similarly for σ, we will have $\check{\tau}_l \leqq \check{\sigma}_l - 1$. Thus $\check{\sigma}_l \neq 0$ and

$$(n-l)(\check{\sigma}_l - 1) \geqq (n-l)\, \check{\tau}_l \geqq (n-l)\, \check{\tau}_{l+1} \geqq \gamma_{l+1}(\tau) \geqq \gamma_{l+1}(\sigma),$$

so $l \in L$. Thus $\gamma_k(\tau) \geqq \gamma_k(\sigma)$ for all $k \in L$ implies $\tau \geqq \sigma$.

To show that L is the unique minimal set with this property, choose an integer $l \in L$. We will construct a diagram $\tau \ngeqq \sigma$ with at most n columns such that $\gamma_k(\tau) \geqq \gamma_k(\sigma)$ for all $k \neq l$.

We will specify τ by determining its column lengths $\check{\tau}_i$. Of course we take $\check{\tau}_i = 0$ for $i > n$. Choose an integer $N \gg 0$ ($\gamma_1(\sigma)+1$ will do), and set $\check{\tau}_i = N$ for $i = 1, \ldots, l-1$.

Write $\gamma_{l+1}(\sigma) = (n-l)\,q - r$, with $0 \leqq q$ and $0 \leqq r < n-l$, and set $\check{\tau}_i = q$ for $i = l, \ldots, n-r$. If $q=0$, then $\gamma_{i+1}(\sigma)=0$ and $r=0$, so $\check{\tau}_i$ is already defined for every i. If $q>0$, then we set $\check{\tau}_i = q-1$ for $i = n-r+1, \ldots, n$.

Since N was large, $\gamma_k(\tau) \geqq \gamma_k(\sigma)$ for $k<l$. By construction,

$$\gamma_{l+1}(\tau) = (n-l)\,q - r = \gamma_{l+1}(\sigma).$$

Generally, suppose that for some $k \geqq l+1$ we have $\gamma_k(\tau) \geqq \gamma_k(\sigma)$. If we had $\gamma_{k+1}(\tau) < \gamma_{k+1}(\sigma)$, then we would have $\check{\tau}_k > \check{\sigma}_k$. But then for $k \leqq j \leqq n$ we would have

$$\check{\tau}_j \geqq \check{\tau}_k - 1 \geqq \check{\sigma}_k \geqq \check{\sigma}_j,$$

so $\gamma_{k+1}(\tau) \geqq \gamma_{k+1}(\sigma)$. Thus $\gamma_k(\tau) \geqq \gamma_k(\sigma)$ for all $k \neq l$.

Finally, note that $\gamma_{l+1}(\tau) = \gamma_{l+1}(\sigma)$ together with $\check{\tau}_l = q \leqq \check{\sigma}_l - 1$ imply $\gamma_l(\tau) < \gamma_l(\sigma)$, so $\tau \ngeqq \sigma$, as required. //

B. Tableaux

Finally we discuss the notion of a (Young) tableau on the numbers $1, \ldots, n$.

If $\sigma \subset \mathbb{N} \times \mathbb{N}$ is a diagram, then a tableau T of shape σ is a map $T: \sigma \to \{1, \ldots, n\}$. We write $\sigma = |T|$, and we think of T as a way of filling in the "boxes of σ" with numbers between 1 and n. For example,

1	3	3
2	1	
3	1	

is a tableau with shape (3, 2, 2). By the i^{th} *row of* T we mean the sequence $T(i,1), T(i,2), \ldots$. Similarly, we speak of the columns of a tableau (= the rows of the dual tableau $\check{T}$), etc. The *content* of a tableau T is the function $C_T: \{1, \ldots, n\} \to \mathbb{N}$ such that $C_T(p)$ is the number of times p occurs in T.

We next define an order on tableaux that extends the order $\leqq$ on diagrams.

If S, T are tableaux (on possibly different diagrams) we write $S \leqq T$ iff for all p, q the first p rows of S contain fewer occurrences of integers $\leqq q$ than do the corresponding rows of T; in symbols

$$\begin{aligned} &\#\{(i,j) \in |S| \mid i \leqq p \quad \text{and} \quad S(i,j) \leqq q\} \\ &\quad \leqq \#\{(i,j) \in |T| \mid i \leqq p \quad \text{and} \quad T(i,j) \leqq q\}. \end{aligned}$$

Lemma 1.5. a) *If $S \leqq T$ are tabelaux, then $|S| \leqq |T|$.*

b) *Suppose that S and T are tabelaux having only one row, say $|S| = (k)$ and $|T| = (l)$. Suppose further that the elements of these rows are arranged in increasing* *order, that is* $S(1,i) \leqq S(1,i+1)$ *and* $T(1,i) \leqq T(1,i+1)$ *for appropriate values of* i. *Then* $S \leqq T$ iff $k \geqq l$ *and* $S(1,i) \geqq T(1,i)$ *for* $1 \leqq i \leqq l$. *(Note the change in the direction of the inequality! In earlier versions of this paper we wrote* $S \trianglelefteq T$ *for this case of* $S \geqq T$.*)*

Proof. a) For large q, the number of integers $\leqq q$ in the first p rows of S is $\beta_p(|S|)$, and similarly for T. Thus $S \leqq T$ implies $\beta_p(|S|) \leqq \beta_p(|T|)$ for all p.

b) With the given hypothesis, the number of $S(1,i)$ that are $\leqq p$ is the greatest i with $S(1,i) \leqq p$. The number of $T(1,j) \leqq p$ will be at least i iff $T(1,i) \leqq p$. //

The following notion is fundamental:

A tabelaux T is called *Standard* if its rows are strictly increasing sequences and its columns are non-decreasing sequences; that is, if $T(i,j) < T(i,j+1)$ and $T(i,j) \leqq T(i+1,j)$ whenever these inequalities make sense. Equivalently, T is standard if its rows are strictly increasing sequences and if denoting the i^{th} row, considered as a tableau of one row, by T_i, we have $T_i \geqq T_{i+1}$ for each i.

A *double (standard) tabelaux* is a pair $(S|T)$ of (Standard) tableaux, with $|S| = |T|$. Following [Doubilet-Rota-Stein] we use these to indicate products of minors of a matrix; if $X = (X_{ij})$ is an $n \times m$ matrix with $n \leqq m$, and if $|S| = |T|$ has at most n columns, then we associate to $(S|T)$ the product of minors of X whose i^{th} factor is the minor involving rows $S(i,1)$, $S(i,2), \ldots$ and columns $T(i,1)$, $T(i,2), \ldots$. Thus the i^{th} factor is a minor of order $|S|_i$, the length of the i^{th} row of $|S|$. When we wish to write out a double standard tableau $(S|T)$ with $|S| = \sigma = (\sigma_1, \ldots, \sigma_l)$, we will write

$$(S|T) = \left(\begin{array}{r|l} s_{1\sigma_1}, \ldots\ldots, s_{11} & t_{11}, \ldots\ldots, t_{1\sigma_1} \\ s_{2\sigma_2}, \ldots, s_{21} & t_{21}, \ldots, t_{2\sigma_2} \\ \cdots & \\ s_{l\sigma_l}, \ldots, s_{l1} & t_{l1}, \ldots, t_{l\sigma_l} \end{array} \right),$$

to mean that $S(i,j) = s_{ij}$ and $T(i,j) = t_{ij}$. *Note that we have written the rows of* S *backwards.*

We will partially order the double tableaux (or the corresponding products of minors of X) by setting

$$(S|T) \leqq (S'|T')$$

if $S \leqq S'$ and $T \leqq T'$. The *content* $C_{(S|T)}$ of $(S|T)$ is by definition the pair (C_S, C_T).

If σ is a diagram of $\leqq n$ columns then there are two special standard tableaux of shape σ: the *canonical* tableau C_σ whose i^{th} row is $(1, 2, \ldots, \sigma_i)$ and the *anticanonical* tableau $\bar{C}_\sigma$ whose i^{th} row is $(n - \sigma_i + 1, \ldots, n-1, n)$. These are respectively the last and first in the ordering $\leqq$ among the tableaux of shape σ. For example,

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & \\ \cline{1-2} 1 & 2 & \\ \cline{1-2} \end{array} \quad \text{is canonical,}$$

while if $n=5$,

$$\begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 4 & 5 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} \end{array} \quad \text{is anticanonical.}$$

If no integer occurs twice in one row of a tableau T, then there is a unique way of permuting the elements of each row to make the rows strictly increasing.

We will describe a systematic way of changing a standard tableau into the anticanonical tableau of the same shape, which will play an important role in the sequel:

For $i \neq j$, consider the operation S_i^j on a standard tableau T that changes each row of T that contains i but not j by replacing i by j, and then rearranging the row so that it is again in strictly increasing order.

We write $h_i^j(T)$ for the number of rows of T that contain i but not j; equivalently,

$$\begin{aligned} h_i^j(T) = &(\text{The number of rows of } T \text{ containing } i) \\ &-(\text{The number of rows of } S_i^j(T) \text{ containing } i). \end{aligned}$$

For example,

$$S_1^4 \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}, \qquad h_1^4 = 1.$$

Note that in this example, T is standard but $S_i^j(T)$ is not. However, we have

Proposition 1.6. *Let $1 \leqq i \leqq j \leqq n$ be integers. If T is a standard tableau on $1, \ldots, n$ with the property that if a row of T contains an integer $k \leqq i$ then it contains all the integers $i, i+1, \ldots, j-1$; then $S_i^j(T)$ is again standard and T is determined by $S_i^j(T)$ together with $h_i^j(T)$.*

Proof. From the hypothesis it follows that the rows of T not containing i must all follow the rows of T containing i. Thus T is obtained from $S_i^j(T)$ by replacing j by i in the first $h_i^j(T)$ rows that contain j but not i.

To show that $S_i^j(T)$ is standard, it suffices to consider the case where T has two rows. The action of S_i^j on a row satisfying the hypothesis can at most replace the entry p with $p+1$ for $i \leqq p < j$. Thus the only possible violation of standardness in $S_i^j(T)$ arises in the case in which i is in the first row of T, and, for some p with $i \leqq p < j$, p occurs in the same position in both the first and second rows.

Since the first row must contain the sequence $i, i+1, i+2, \ldots, p$, and since the sequence of elements in the second row is strictly increasing and T is standard, i must occur in the second row, and must be in the same position as in the first row. If j does not occur in the second row of T, then the p in the second row will also be replaced by $p+1$, and standardness will be preserved. But if j occurs in the second row, then it must occur exactly $j-i$ positions after i, since all the integers $i+1,\dots,j-1$ must also occur there, by hypothesis. However, the first row must have an element $>j$ in the position $j-i$ positions after i, since it too contains $i+1,\dots,j-1$, but does not contain j. This contradicts the standardness of T.

We can pass from any tableau to an anticanonical tableau by applying the composite operator

$$S^n_{n-1}\,S^n_{n-2}\,S^{n-1}_{n-2}\dots S^3_1\,S^2_1.$$

If we write h^j_i for the number of substitutions of j for i made by the application of S^j_i in this sequence, we have

Corollary 1.7. *A standard tableau T is determined by its shape together with the nmbers h^j_i defined above.*

Proof. This follows from Proposition 1.6 by an easy induction.

2. The Straightening Formula

The straightening formula expresses any monomial in the minors of a matrix $X=(X_{ij})$ (a double tabelau) as an integral linear combination of monomials that are later in the order $\leqq$ of section 1B, and leads to a unique expression for any double tableau as an integral linear combination of double standard tableaux. In this section we present a new proof of this result, found in joint work with Seshadri; essentially, we first give a new proof of a "straigthening formula" discovered by Hodge for the homogeneous coordinate ring of a Grassmann variety, and then deduce the version for polynomial rings by passing to an affine open subset. Our statement of the straightening formulas contain an improvement on the originals suggested to us by Hesselink and Stein. Aside from the originals [Doubilet-Rota-Stein] and [Hodge], two other proofs of the Straightening formula may be found in [Kung-Rota] and [Akin-Buchsbaum-Weyman]. Our treatment has the advantage (for us) of being linear algebra instead of combinatorics, and of doing the hard part of the work in the relatively simpler setting of the Grassmannian. For an axiomatization of the notion of straightening law, and many other examples, see [DeConcini-Eisenbud-Procesi].

We suppose as in the introduction that $X=(X_{ij})$ is an $n\times m$ matrix of indeterminates, with $n\leqq m$. We write $R=F[X_{ij}]$ for the polynomial ring in all the X_{ij}. We will deal throughout this section only with tableaux and double tableaux whose shapes have at most n columns, and we will use the words "double tableaux" as in section 1B to refer to the monomials in the minors of X.

The Straightening formula is, for each nonstandard double tableau $\mathcal{M}\in R$ an expression of the form

$$\mathcal{M}=\Sigma\, n_i\,\mathcal{M}_i \qquad (*)$$

with $n_i\in Z$, $\mathcal{M}_i>\mathcal{M}$, and $C_{\mathcal{M}_i}=C_{\mathcal{M}}$.

Since there are only finitely many double tableaux of a given content, an obvious induction shows that every double tableau is a linear combination of double standard tableaux. But one has more:

Theorem 2.1 (Doubilet-Rota-Stein). *The double standard tableaux form an F-free bases for the polynomial ring $R=F[X_{ij}]$. Moreover, if*

$$\mathscr{M}=\Sigma f_i \mathscr{M}_i, \quad 0 \neq f_i \in F$$

is the unique expression of a double tableau $\mathscr{M}$ as a linear combination of distinct double standard tableaux then $\mathscr{M}_i \geqq \mathscr{M}$ and $C_{\mathscr{M}_i}=C_{\mathscr{M}}$ for each i.

We will deduce (*) and 2.1 from similar results on the homogeneous coordinate ring of the grassman variety of n-dimensional subspaces of an $m+n$-dimensional space. To do this, we introduce an $n \times m+n$ matrix $Y=(Y_{ij})$ whose entires Y_{ij} are new indeterminates. We denote by R' the subring of $F[Y_{11}, \dots, Y_{n,m+n}]$ generated by the $n \times n$ minors, called Plücker coordinates, of the matrix Y; this is the homogeneous coordinate ring in question.

We denote a Plücker coordinate by giving, in increasing order, the indices of the columns involved, written in square brackets. Thus if $1 \leqq a_1 < \dots < a_n \leqq m+n$, then $[a_1, \dots, a_n]$ is a Plücker coordinate. If p is a permutation of $\{1, \dots, n\}$, then we agree that $[p(a_1), \dots, p(a_n)]$ represents $(-1)^{\mathrm{sgn}(p)}[a_1, \dots, a_n]$. We think of this Plücker coordinate as an alternating multilinear functional on the vectors whose coordinates are

$$(Y_{a_1 1}, \dots, Y_{a_1 n}), \dots, (Y_{a_m 1}, \dots, Y_{a_n n}).$$

We associate to any monomial in the Plücker coordinates, $[a_{11} \dots a_{1n}]\,[a_{21} \dots a_{2n}] \dots$, the tableau of shape $(n, n, \dots)$ whose i^{th} row is $a_{i1}, \dots, a_{in}$. We order these monomials by using the order $\leqq$ on tableaux, and we say that a monomial is *standard* if the corresponding tableau is standard. We will sometimes write $[T] \in F[Y_{ij}]$ for the monomial in the Plücker coordinates, corresponding to a tableau T.

We claim first that, for any two Plücker coordinates $[a_1, \dots, a_n]$ and $[b_1, \dots, b_n]$ and any integer k between 1 and n, we have

$$\sum_p (-1)^{\mathrm{sgn}\, p}[a_1, \dots, a_{k-1}, p(a_k), \dots, p(a_n)] \cdot [p(b_1), \dots, p(b_k), b_{k+1}, \dots, b_n] = 0$$

where the sum extends over all permutations p of the symmetric group

$$S=\mathrm{Symm}\,\{a_k, \dots, a_n, b_1, \dots, b_k\}.$$

This is because the sum on the left hand side represents an alternating multilinear functional on the $n+1$ vectors with coordinates

$$(Y_{a_k 1}, \dots, Y_{a_k n}), \dots, (Y_{b_k 1}, \dots, Y_{b_k n}).$$

Since these are n-dimensional vectors, any such functional is 0.

Next note that because of the alternating character of the Plücker coordinates, all the terms in the above sum that correspond to elements of a given coset of the subgroup

$$S_1=\mathrm{Symm}\,\{a_k, \dots, a_n\} \times \mathrm{Symm}\,\{b_1, \dots, b_k\} \subset S$$

are the same. Thus in the equality above we may restrict the summation to coset representatives of S_1 in S. (Note that it suffices to prove the identity for $F=Z$, the ring of integers, where integers are nonzerodivisors.)

Finally, suppose that the product $[a_1, \ldots, a_n](b_1, \ldots, b_n]$ is not standard, so that for some k we have

$$b_1 < \ldots < b_k < a_k < \ldots < a_n.$$

Using this k, the above equality can be written

$$\begin{aligned}[][a_1, \ldots, a_m][b_1, \ldots, b_m] \\ = -\sum_{p \in S/S_1} (-1)^{\operatorname{sgn} p} [a_1, \ldots, a_{k-1}, p(a_k), \ldots, p(a_m)] \qquad (*)' \\ \cdot [p(b_1), \ldots, p(b_k), b_{k+1}, \ldots, b_m], \quad p \text{ not the identity coset}\end{aligned}$$

and, in this formula, all the products on the right are seen to be later in the ordering $\leqq$ than the products on the left. This is the "straightening formula for Plücker coordinates". An easy induction proves, as before, that every product of Plücker coordinates can be represented as a linear combination of standard products of Plücker coordinates.

Here, again, we also have independence:

Theorem 2.1′ ([Hodge]). *The standard products of Plücker coordinates form an F-free basis for R′. Moreover, if we express any product $\mathcal{M}$ of Plücker coordinates in terms of this basis, $\mathcal{M} = \Sigma f_i \mathcal{M}_i$, with $\mathcal{M}_i$ standard and distinct and $0 \neq f_i \in F$, then $\mathcal{M}_i \geqq \mathcal{M}$, and $c_{\mathcal{M}_i} = c_{\mathcal{M}}$, for each i.*

Proof. In view of $(*)'$, it is enough to prove the linear independence of the standard products. For $\lambda \in R$, and $i \neq j$, let $E_i^j(\lambda) \in GL(n+m, F)$ be the matrix with ones on the diagonal, λ in the $(j, i)^{\text{th}}$ place, and zeros elsewhere.

The group $GL(n+m, F)$ acts on the space of $n \times n+m$ matrices, and thus on the elements of $F[Y_{ij}]$, thought of as functions on this space; if $E \in GL(n+m, F)$, $\phi \in F[Y_{ij}]$, and M is a matrix, then $(E\phi)(M) = \phi(ME)$. Thus

$$\begin{aligned} E_i^j(\lambda)[a_1, \ldots, i, \ldots, a_n] &= [a_1, \ldots, i, \ldots, a_n] \\ &\quad + \lambda [a_1, \ldots, \hat{i}, j, \ldots, a_n], \end{aligned}$$

where $\hat{i}$ means that i is omitted. More generally for any tableau A we will have, as in the notation of section 1B,

$$E_i^j(\lambda)[T] = \lambda^{h_i^j(T)}[S_i^j T] + \text{terms of lower degree in } \lambda.$$

Now suppose that $\sum_k f_k [A_k] = 0$, where the A_k are standard and distinct and $f_k \neq 0$. If t is a new indeterminate, then over $F' = F(t)$ we will have

$$\sum f_k E_i^j(t)[A_k] = 0. \qquad (\#)$$

Suppose now that, for some $1 \leqq i < j \leqq n+m$, all the A_k satisfy the hypothesis of Proposition 1.6. By virtue of that proposition, the coefficient of the highest power of t in (#) will have the form

$$\sum f_k [S_i^j A_k],$$

where the sum is taken over some subset of the indicates k, and where the $S_i^j A_k$ are again distinct and standard, with $f_k \neq 0$. Applying this technique sequentially for the pairs of indices $(1,2), (1,3), \ldots, (n+m-1, n+m)$ as in Corollary 1.7, we get a nontrivial relation among the products of Plücker coordinates represented by anti-canonical tabelaux. Since these are all of the form $[m+1, \ldots, n+m]^h$, for various powers h, this is absurd; for example, they different degrees as polynomials in X_{ij}.

Remark. Exactly the same technique could be used to prove the linear independence statement of Theorem 2.1 directly; the necessary ideas are developed in Sect. 3.

To establish the straightening formula (*) and to prove 2.1, we specialize Y to the matrix

$$X' = \begin{pmatrix} X_{11} & \ldots & X_{1m} & 0 & \ldots & 0 & 1 \\ \vdots & & & \vdots & & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & & & 0 & \cdot^{\cdot^{\cdot}} & & \vdots \\ X_{n,1} & \ldots & X_{nm} & 1 & 0 & \ldots & 0 \end{pmatrix} = (X, E)$$

which is essentially X followed by an $n \times n$ identity matrix E. (We take the identity E matrix in the inverted form above to make our choice of orderings match; see Lemma 2.2.) This specialization corresponds to a map $R' \to R$, carrying Plücker coordinates in R' to $n \times n$ minors of X'.

Remark. The given ordering of Plücker coordinates defines a cellular decomposition for the Grassman variety (the Bruhat decomposition in Schubert cells). The oposite order also gives rise to a cellular structure for whcih Spec R is (via the map $R' \to R$) the open cell. The first cellular subdivision induces on Spec R a decomposition corresponding to the ordering of minors.

The $k \times k$ minor of X represented by $(a_k, \ldots, a_1 | b_1, \ldots, b_k)$ is equal up to sign to the $n \times n$ minor of X' involving columns $b_1, \ldots, b_k, \alpha_{k+1}, \ldots, \alpha_n$, where $\alpha_{k+1} < \ldots < \alpha_n$ and $\{\alpha_{k+1}, \ldots, \alpha_n\}$ is obtained from the complement $\{1, \ldots, n\} - \{a_1, \ldots, a_k\} = \{\bar{a}_1, \ldots, \bar{a}_{n-k}\}$ of $\{a_1, \ldots, a_k\}$ in $\{1, \ldots, n\}$ (with the $\bar{a}_i$ arranged so that $\bar{a}_1 < \ldots < \bar{a}_{n-k}$, say) by

$$\{\alpha_{k+1}, \ldots, \alpha_n\} = \{m+n-\bar{a}_{n-k}+1, \ldots, m+n-\bar{a}_1+1\}.$$

The map $(a_k, \ldots, a_1 | b_1, \ldots, b_k) \mapsto [b_1, \ldots, \alpha_n]$ establishes a $1-1$ correspondence between the minors of X and the Plücker coordinates of X'. We think of the 0-order minor $1 = (1)$ of X as corresponding to $[m+1, \ldots, m+n]$.

Lemma 2.2. *Under the above correspondence, the orderings $\leqq$ correspond. In particular, standard monomials in the minors of X are the same as standard monomials in the Plücker coordinates of X' not involving $[m+1, \ldots, m+n]$.*

Proof. If $(a|b) = (a_k, \ldots, a_1 | b_1, \ldots, b_k) \to [b_1, \ldots, \alpha_n] = P$ and $(a'|b') = (a'_l, \ldots, a'_1\, b'_1, \ldots, b'_l) \to [b'_1, \ldots, \alpha'_n] = P'$ then we must check that $(a'|b') \leqq (a|b)$ iff $P' \leqq P$. A necessary condition for the first of these is $l \leqq k$. But each $b'_i \leqq n$ and $\alpha_j \geqq n$ for each i, j. If we had $k \leqq l$, then $P' \leqq P$ would entail the inequality $b'_k \geqq \alpha_k$, a contradiction; thus $l \leqq k$ is necessary for either of our inequalities.

We thus may suppose $l \leqq k$. If in fact $l = k$, then the only thing to prove is that $a'_i \geqq a_i$ for $i = 1, \ldots, k$ iff $\alpha'_j \geqq \alpha_j$ for $j = k+1, \ldots, n$. For this we note that

$$(a_1, \ldots, a_k) \to (\bar{a}_1, \ldots, \bar{a}_{n-k})$$

is an order reversing transformation from the set of tableaux of shape (k) on $\{1, \ldots, n\}$ to the set of tableaux of shape $n-k$, both with the order $\leqq$. Since

$$(\bar{a}_1, \ldots, \bar{a}_{n-k}) \to (m+n-\bar{a}_{n-k}+1, \ldots, m+n-\bar{a}_1+1) = (\alpha_{k+1}, \ldots, \alpha_n)$$

is also order reversing (this time from the set of tableaux of shape $(n-k)$ to itself), the case $k = l$ follows.

If on the other hand $k > l$, we may assume by induction that the lemma has been verified for k and $l+1$, and we reduce to this case. Let c and d be respectively the greatest elements in $\{1, \ldots, n\} - \{a'_1, \ldots, a'_l\}$ and $\{1, \ldots, m\} - \{b'_1, \ldots, b'_l\}$, and let a'' and b'' be the tableaux of shape $(l+1)$ the elements of whose single rows are obtained, respectively, by putting the sets $\{a'_1, \ldots, a'_l, c\}$ and $\{b'_1, \ldots, b'_l, d\}$ into strictly increasing order. The tableaux a'' is, with respect to $\leqq$, the unique minimal tableau of one row whose row has length $> l$ which is $\geqq a'$, so $a \geqq a'$ iff $a \geqq a''$, and similarly for b. Also, our correspondence sends

$$(a'' \mid b'') \mapsto [b''_1, \ldots, b''_{l+1}, \alpha'_{l+2}, \ldots, \alpha'_n] = P''.$$

Since in any case $\alpha'_l > b_l$, the minimality property of b implies that $P' \leqq P$ iff $P'' \leqq P$, so we are done.

We see from the lemma that $(*)$ follows from $(*')$; this completes our proof of the straightening formula.

It remains to deduce Theorem 2.1 from 2.1′. To this end, we set $\bar{R}' = R'/(\varepsilon - [m+1, \ldots, m+n])$, where

$$\varepsilon = (-1)^{\frac{m(m-1)}{2}} = \det E.$$

We claim that the standard monomials in the $n \times n$ minors of Y that do not have $[m+1, \ldots, m+n]$ as a factor form an F-free basis of $\bar{R}'$. Since they obviously span, it is enough to show they are linearly independent; that is we must show that in R' there is no equality of the form

$$\sum f_i \mathscr{M}_i = [m+1, \ldots, m+n] \left(\sum f'_i \mathscr{M}'_i\right),$$

with $0 \neq f_i$, $f'_i \in F$, the $\mathscr{M}_i$ standard and distinct, not involving $[m+1, \ldots, m+n]$ and the $\mathscr{M}'_i$ standard and distinct. But $[m+1, \ldots, m+n]$ is the earliest Plücker coordinate in the ordering $\leqq$, so the monomials $[m+1, \ldots, m+n]\ \mathscr{M}'_i$ are again standard, distinct, and distinct from the $\mathscr{M}_i$. The impossibility of the relation above now follows from the linear independence of the standard monomials in R'.

To finish the proof of Theorem 2.2, we will show that the specialization $Y \to X'$ carries R' onto R with kernel $I = (\varepsilon - [m+1, \ldots, m+n])$, thus identifying $\bar{R}'$ with R. Certainly the map takes $[m+1, \ldots, m+n]$ to ε. It is onto since the $n \times n$ minors of X' generate R. Finally, it is enough to show that the kernel is no larger than I for $F = Z$, the ring of integers; the general result is then obtained by tensoring with F. But $\dim \bar{R}' = \dim R$, and since R' is a domain, $\bar{R}'$ is too (an affine open subset of an irreducible projective variety is irreducible!) Thus the map $\bar{R} \to R$ can have no kernel. Alternatively, as Hesselink has remarked, one can check that if, over some ring, an $n \times n$ matrix B satisfies $\det B = \det E$, then for any $n \times m$ matrix A the matrix $(A\ B)$ has the same minors as the matrix $(EB^{-1}A\ E)$; this yields an elementary proof of the isomorphism.

It will be useful to have $(*)$ in a slightly more explicit form, at least for the product of two minors of X. Suppose

$$\mathscr{M} = \begin{pmatrix} a^1_{\sigma_1}, \ldots, a^1_1 & b^1_1, \ldots, b^1_{\sigma_1} \\ a^2_{\sigma_2}, \ldots, a^2_1 & b^2_1, \ldots, b^2_{\sigma_2} \end{pmatrix}$$

has $\sigma_2 \leqq \sigma_1$ but is non-standard in R. Say there is an integer k such that $a^1_k > a^2_k$. The corresponding sum $(*)'$ has $k \leqq n$ and has two kinds of terms (in the notation of $(*)'$): 1) those for which $p(b_1), \ldots, p(b_k)$ are all $\leqq n$, and those 2) for which some of the $p(b_i)$ are $> n$. Corresponding to these two types of terms we write

$$\mathscr{M} = \Sigma^{1)} \pm \mathscr{M}^{(1)}_i + \Sigma^{2)} \pm \mathscr{M}^{(2)}_i. \qquad (**)$$

The $\mathscr{M}^{(1)}_i$ are all monomials with the same shape as $\mathscr{M}$; they are obtained from $\mathscr{M}$ by exchanging a certain number p of the elements $\{a^2_1, \ldots, a^2_k\}$ with the same number of the elements of $\{a^1_k, \ldots, a^1_{\sigma_1}\}$, in all possible ways and for all p. We indicate this term symbolically by

$$\sum \pm \begin{pmatrix} \underline{a^1_{\sigma_1}, \ \ldots, a^1_k}, \ a^1_{k-1}, \ldots, a^1_1 & b^1_1, \ \ldots \ b^1_{\sigma_1} \\ \underline{a^2_{\sigma_2} \ldots, a^2_k}, \quad \ldots \quad a^2_1 & b^2_1, \ldots, b^2_{\sigma_2} \end{pmatrix}$$

underlying the affected numbers.

The terms $\mathscr{M}^2_i$ on the other hand are obtained from $\mathscr{M}$ by exchanging a certain number p of the $\{a^2_1, \ldots, a^2_k\}$ for a smaller number q of the a^1_i, and at the same time removing $p-q$ of the b^2_i from the second row, and adjoining them to the row of b^1_i in all possible ways, for all $0 \leqq q < p \leqq k$. (Of course if the rows produced this way have repeated elements, the corresponding minors are 0.) In particular, σ_1 is made larger, and the terms $\mathscr{M}^{(2)}_i$ have shapes that are $>$ the shape of $\mathscr{M}$. There is of course a symmetric equation involving column indices.

For example, if $\mathscr{M} = (41|12)(32|34)$, or in double tableau notation

$$\mathscr{M} = \begin{pmatrix} 41 & 12 \\ 32 & 34 \end{pmatrix},$$

and $k = 2$, then the first sum is

$$\begin{pmatrix} \underline{4}1 & 12 \\ \underline{32} & 34 \end{pmatrix} = \pm \begin{pmatrix} 21 & 12 \\ 43 & 34 \end{pmatrix} \pm \begin{pmatrix} 31 & 12 \\ 42 & 34 \end{pmatrix},$$

while the second sum has terms

for $p=1$, $q=0$:

$$\pm\left(\begin{array}{r|l}431 & 123\\ 2 & 4\end{array}\right)\pm\left(\begin{array}{r|l}431 & 124\\ 2 & 3\end{array}\right)\pm\left(\begin{array}{r|l}421 & 123\\ 3 & 4\end{array}\right)\pm\left(\begin{array}{r|l}421 & 124\\ 3 & 3\end{array}\right);$$

for $p=2$, $q=1$:

$$\pm\left(\begin{array}{r|l}321 & 123\\ 4 & 4\end{array}\right)\pm\left(\begin{array}{r|l}321 & 124\\ 4 & 3\end{array}\right);$$

for $p=2$, $q=0$:

$$\pm(4321\,|\,1234).$$

It happens that these are all standard.

As a first immediate application of Theorem 2.1, we define a filtration of R by ideals which play an important role in the sequel.

For each diagram σ, let A_σ be the F-linear span in R of the double tableaux (standard or not) of shape $\geqq\sigma$. We have:

Corollary 2.3. 1) *A_σ is an ideal of R.*

2) *A_σ has a basis consisting of the double standard tableaux of shape $\geqq\sigma$.*

Proof. For 1), note that if $\mathcal{M}$ is a monomial (double tableau) of shape $\geqq\sigma$ and $\mathcal{N}$ is a monomial of shape τ then $\mathcal{M}\mathcal{N}$ has shape $\geqq\sigma\tau\geqq\sigma$, so $\mathcal{M}\mathcal{N}\in A_\sigma$. Part 2) follows from Theorem 2.1 using Lemma 1.5a. //

3. Representation Theory

Keeping the notation of the introduction, we next turn to the study of R as a G-module. Since any element of G takes a $k\times k$ minor of X into a linear combination of $k\times k$ minors of X, we have immediately:

Lemma 3.1. *If σ is a diagram, then the space spanned by the double tableaux of shape σ is a G-module, as is the space A_σ spanned by the double tableaux of shape $\geqq\sigma$.*

Write $A'_\sigma\subset A_\sigma$ for the space spanned by the double tableaux of shape $\gneqq\sigma$. We next analyze the G-module A_σ/A'_σ.

Given tableaux A and B of the same shape, we write $(A|B)$ for the double tableau formed from A and B. As in Sect. 1, we write C_σ and $\bar{C}_\sigma$ for the canonical and anticanonical tableaux of shape σ. We say that a double tableau is right (respectively left) semi-canonical if it has the form $(A|C_\sigma)$ (respectively $(C_\sigma|B)$), and we write K_σ and $\bar{K}_\sigma$ for $(C_\sigma|C_\sigma)$ and $(\bar{C}_\sigma|\bar{C}_\sigma)$ respectively.

Let L_σ and ${}_\sigma L$ be the spaces spanned respectively by all the right and left semicanonical tableaux of shape σ. By symmetry, it is enough to treat L_σ. It is easy to see that L_σ is a $GL(n,F)$-submodule of R. The importance of this construction is shown by the following result:

Theorem 3.2. 1) *L_σ has a basis consisting of the double standard tableaux of the form $(A|C_\sigma)$.*

2) *There is a G-isomorphism*

$$L_\sigma \otimes_{F\,\sigma} L \to A_\sigma / A'_\sigma$$

which, for any tableaux A and B of shape σ, takes

$$(A|C_\sigma) \otimes (C_\sigma|B) \to (A|B).$$

Proof. 1) We use the explicit form (**) of the straightening formula given in Sect. 2 to express any double tableau $(A|C_\sigma)$ in terms of standard ones. To do this we successively apply (**) to pairs of adjacent rows of $(A|C_\sigma)$ and to pairs of adjacent rows in the double tableaux thus produced. Since each row of C_σ is a subsequence of the preceding row, the terms of the form $\mathcal{M}_i^{(2)}$ of (**) never arise, and the straightening formula finally expresses $(A|C_\sigma)$ as a linear combination of double standard tableaux of the same shape. This proves 1).

2) It suffices to show that the given map is well-defined, since the given formula, applied only for double standard tableaux $(A|B)$ gives an F-linear isomorphism, by 1). For an arbitrary double tableau $(A|B)$ of shape σ we may write $(A|C_\sigma) = \sum f_i(A_i|C_\sigma)$ and $(C_\sigma|B) = \sum g(C_\sigma|B_i)$ with A_i and B_i standard, as above. Using (**) as above one sees that $(A_i|B_j) = \sum f_i g_j (A_i|B_j) + E$, where E is a combination of standard monomials of shape $\gneqq \sigma$. Thus $(A|B) = \sum f_i g_j (A_i|B_j) \bmod A'_\sigma$, as required.

We next turn to the case in which F is an infinite field. We let $U^+(n,F) \subset GL(n,F)$ be the subgroup of upper triangular matrices whose diagonal entries are all equal to 1.

Theorem 3.3. *Any $U^+(n,F)$-submodule of R contains some nonzero linear combination of elements of the form $(\bar{C}_\sigma|B)$. In particular,*

0) *The invariants of $U^+(n,F)$ are spanned by elements of the form $(\bar{C}_\sigma|B)$.*

1) *Any $U^+(n,F)$ submodule of L contains $(\bar{C}_\sigma|C_\sigma)$.*

2) *L_σ is $U^+(n,F)$-indecomposable, and, a fortiori, $GL(n,F)$-indecomposable.*

3) *If the characteristic of F is 0, then L_σ is $GL(n,F)$-irreducible.*

Proof. If $1 \leqq i < j \leqq n$, and $\lambda \in F$, we write $E_i^j(\lambda)$ for the element of $U^+(n,F)$ with λ in the $(i,j)^{\text{th}}$ position and all other entries above the diagonal 0. If X is any $n \times m$ matrix, then $E_i^j(\lambda)X$ is the matrix obtained by adding λ times the j^{th} row of X to the i^{th} row. If we apply $E_i^j(\lambda)$ to the minor $m = (a_k, \ldots, a_1 | b_1, \ldots, b_k)$, regarded as a functional on $n \times m$ matrices, we obtain $E_i^j(\lambda)m = m$ if i does not occur or if both i and j occur among $a_1, \ldots, a_k$. Otherwise, $E_i^j(\lambda)m = m - \lambda m'$, where m' is obtained from m by replacing i by j among the $a_1, \ldots, a_k$. If $(A|B)$ is a double tableau then, thinking of λ as a variable, we may write $E_i^j(\lambda)(A|B)$ as a polynomial in λ; in the notation of Proposition 1.6 we will have:

$$E_i^j(\lambda)(A|B) = \lambda^{h_i^j(A)}(S_i^j A|B) + (\text{terms of lower degree in } \lambda).$$

Now if $W \subset V$ are vector spaces over an infinite field F, and if for some $v_0, \ldots, v_k \in V$ and all $\lambda \in F$ we have $\sum_0^h \lambda^i v_i \in W$, then $v_i \in W$ for every i. Thus if we apply $E_i^j(\lambda)$ to an element

$$\alpha = \sum f_k(A_k | B_k),$$

such that the maximum value of $h_i^j(A_k)$ is h, then

$$\alpha' = \sum_{h_i^j A_k = h} f_k(S_i^j A_k | B_k) \in U^+(n, F)\, \alpha.$$

In general, α' may be 0, but if all the f_k are nonzero, all the $(A_k|B_k)$ are standard and distinct, and all the A_k satisfy Proposition 1.6, then by Proposition 1.6, $\alpha' \neq 0$. Starting with $(i, j) = (1, 2)$, and continuing as in the argument of Corollary 1.7, we see that a nonzero linear combination of $(\bar{C}_{\sigma_k}|B_k)$ is in $U^+(n, F)\,\alpha$. This proves the first statement of the theorem and with it 0).

Point 1) follows at once because L_σ is a $U^+(n, F)$-module, while 2) follows at once from 1), and 3) follows from 2) since every indecomposable $GL(n, F)$-module is irreducible in that case.

For us the most important consequence of this theorem is:

Corollary 3.4. *Let F be an infinite field. Then:*

1) *If $(A|B)$ is a double tableau of shape σ, then the G-module generated by $(A|B)$ contains $K_\sigma = (C_\sigma | C_\sigma)$ and $\bar{K}_\sigma = (\bar{C}_\sigma | \bar{C}_\sigma)$.*

2) *If the characteristic of F is 0, then $L_\sigma \otimes_F {}_\sigma L$ is an irreducible G-module, and $A_\sigma = A'_\sigma \oplus M_\sigma$, where $M_\sigma \cong L_\sigma \otimes_F {}_\sigma L$ is the G-module generated by any semi-canonical (or semi-anti-canonical) tableau of shape σ, that is, $M_\sigma = GK_\sigma \supset L_\sigma$. Furthermore, $R = \oplus_\sigma M_\sigma$, the direct sum over the distinct diagrams whose rows have length $\leqq n$.*

Proof. 1) By Theorem 3.3, we have $(\bar{C}_\sigma|B) \in U^+(n, F)(A|B) \subset G(A|B)$. Since $GL(n)$ contains a permutation matrix reversing the order of the rows, $(C_\sigma|B) \in G(A|B)$ as well. Applying the same argument to columns, we see that $K_\sigma \in G(A|B)$. The element $\bar{K}_\sigma$ arises similarly.

2) If M_1, M_2 are finite-dimensional vector spaces over a field F which are representations of two algebras A_1, A_2, respectively, and irreducible over any extension of F, then it is easy to show from the theory of semi-simple algebras that $M_1 \otimes_F M_2$ is irreducible over $A_1 \otimes_F A_2$. Thus, using Theorem 3.3, $L_\sigma \otimes_F {}_\sigma L$ is irreducible. Since G is reductive, we must have $A_\sigma = A'_\sigma \oplus M_\sigma$ for some M_σ, and $M_\sigma \cong L_\sigma \otimes_F {}_\sigma L$ by Theorem 3.2. We will show that $L_\sigma \subset M_\sigma$, and thus that any right semi-canonical tableau generates M_σ. The other cases are symmetric.

To this end, note that if $L_\sigma \not\subset M_\sigma$, the projection $A_\sigma = A'_\sigma \oplus M_\sigma \to A'_\sigma$ yields a nonzero $GL(n, F)$-linear map $L_\sigma \to A'_\sigma$. But A'_σ is a direct sum of $M_{\sigma'}$ for $\sigma' > \sigma$ by the above argument, and, since $M_{\sigma'} \cong L_{\sigma'} \otimes L_{\sigma'}$, we see that as a $GL(n)$-module A'_σ is a sum of copies of $L_{\sigma'}$, for $\sigma' > \sigma$. It is now enough to check that $L_{\sigma'} \ncong L_\sigma$ for any $\sigma' \neq \sigma$, and this follows at once from a consideration of highest weights: $K_\sigma \in L_\sigma$ spans the space of vectors invariant under $U^-(n, F)$ and its weight relative to the torus T of diagonal matrices is exactly $\check{\sigma}$ (under the identification of the dual of T with Z^n).

The last statement of 2) is obvious.

As a second consequence we recover Igusa's theorem [Igusa].

Corollary 3.5. *Let F be an infinite field. The ring of invariants of R under $SL(n, F)$ has as basis the double standard tableaux of the form*

$$\left(\begin{matrix} n & \dots & 1 \\ n & \dots & 1 \\ & \vdots & \\ n & \dots & 1 \end{matrix}\middle| B\right),$$

the standard products of $n \times n$ minors of X.

Proof. Since $SL(n, F)$ contains $U^+(n, F)$ and also the corresponding group $U^-(n, F)$ of lower triangular unipotents, any invariant under $SL(n, F)$ must also be both a linear combination of double standard tableaux of the form $(C_\sigma|B)$ and of the form $(\bar{C}_{\sigma'}|B)$. But $C_\sigma \neq \bar{C}_{\sigma'}$ unless $\sigma' = \sigma = (n, n, \dots, n)$, as required.

Remark. If $U_{ij}^+(n, F)$ denotes the group generated by the elements $E_1^2(\lambda)$, $E_1^3(\lambda), \dots, E_1^n(\lambda)$, $E_2^3(\lambda), \dots, E_i^j(\lambda)$, then a similar argument shows that the $U_{ij}^+(n, F)$-invariants in R are sums of double standard tableaux of the form $(A|B)$, where, in each row of A, if an integer $h < i$ appears then $h+1, \dots, n$ also appear, and if i appears, then $i+1, \dots, j$ appear.

4. The Classification of G-Invariant Ideals

We assume in this section that F is a field of characteristic 0. Let $M_\sigma \subset R$ be the irreducible submodule defined in Corollary 3.4, and let I_σ be the ideal generated by M_σ. From that Corollary it follows that I_σ is the minimal G-invariant ideal containing K_σ, and that every G-invariant ideal is a sum of ideals of the form I_σ. Of course I_σ must be direct sum of certain modules M_τ, since these are non-isomorphic and irreducible; the next theorem tells us which ones.

Theorem 4.1. $I_\sigma = \sum_{\tau \supseteq \sigma} M_\tau$.

Two immediate consequences of importance are:

Corollary 4.2. 1) *A G-submodule $\sum_{\sigma \in T} M_\sigma$ of R is an ideal if and only if $\sigma \in T$ and $\tau \supseteq \sigma$ imply $\tau \in T$.*

2) *$I_\sigma \subseteq I_\tau$ if and only if $\sigma \supseteq \tau$.*

Remark. We write (1) for the diagram with only one row, having length 1, so that $M_{(1)} = (X_{ij})$, the space of linear forms in R. The proof of Theorem 4.1 amounts (on grounds of homogeneity, for example) to a proof that $M_{(1)} M_\sigma = \sum_{\substack{\tau \supset \sigma \\ \text{and adjacent} \\ \text{to } \sigma}} M_\tau$.

One of the main questions in our theory that remains open is the computation of $M_\sigma M_\tau$ for general σ, τ, and thus the computation of the product $I_\sigma I_\tau$ of G-invariant ideals. One more special case of this is treated below, where we compute at least the product of determinantal ideals.

Proof of Theorem 4.1. We first show that if $\tau \supseteq \sigma$ then $M_\tau \subseteq I_\sigma$. It suffices to treat the case in which τ is adjacent to σ in the $\supseteq$ order, and, by Corollary 3.4, to show that $K_\tau \in I_\sigma$. Clearly, in this case τ differs from σ in a unique row, say the i^{th}, with $\tau_i = \sigma_i + 1$. Expanding $K_\tau = (\tau_i, \ldots, 1 | 1, \ldots, \tau_i)$ along the τ_i^{th} column, we get

$$K_\tau = \sum_{j=1}^{\tau_i} \pm (\tau_i, \ldots, \hat{j}, \ldots, 1 | 1, \ldots, \tau_i)(j | \tau_i),$$

so K_τ is a linear combination of elements of L_σ. Thus $K_\tau \in I_\sigma$, by Corollary 3.4.

Next we show that $I_\sigma \subseteq \sum_{\tau \supseteq \sigma} M_\tau$ by showing that $M_\sigma M_{(1)}$ is contained in $\sum_{\tau \supseteq \sigma} M_\tau$, where $M_{(1)}$ is the space of linear forms of R ((1) represents the diagram with one row, of length 1). We claim first of all that

$$M_\sigma M_{(1)} = G(K_\sigma \bar{K}_{(1)}).$$

To see this, we note that since L_σ has a basis consisting of right-semi-canonical double standard tableaux, it is generated, as a $U^-(n, F)$-module, by K_σ. Thus $L_\sigma \otimes_F {}_\sigma L$ is generated over $U^-(n, F) \times U^+(m, F)$ by $K_\sigma \otimes K_\sigma$, and so, using the isomorphism of Theorem 3.2, $A_\sigma / A'_\sigma \cong M_\sigma$ is generated by K_σ. On the other hand, K_σ is stabilized by $U^+(n, F) \times U^-(m, F)$, while, by symmetry, $M_{(1)}$ is generated over $U^+(n, F) \times U^-(n, F)$ by $\bar{K}_{(1)}$. Thus we may apply the following:

Lemma 4.4 *If $H \leqq G$ are groups, M, N RG-modules, $u \in M$ and $v \in N$ elements such that*

$$M = RG\,u,$$

$$N = RH\,v,$$

$$H\,u = u$$

then

$$RG\,u \otimes v = M \otimes N.$$

Proof.

$$\begin{aligned} RG\,u \otimes v &= RG\,RH\,u \otimes v \\ &= RG(u \otimes RH\,v) \\ &= RG\,u \otimes RH\,v \\ &= M \otimes N. \end{aligned}$$

(As a special case, the tensor product of two irreducible modules over a reductive group is cyclic, being generated by the tensor product of an element of highest weight and an element of lowest weight; in the above lemma, we take H to be the unipotent radical of the Borel subgroup corresponding to the ordering on weights). It remains to show that $K_\sigma \bar{K}_1 \in \sum_{\tau \supseteq \sigma} M_\tau$. We will show that the

$GL(n, F)$-module generated by $K_\sigma \bar{K}_1$ is contained in a sum of $GL(n, F)$-modules isomorphic to L_τ with $\tau \supseteq \sigma$. Since M_τ contains all $GL(n, F)$-submodules of R isomorphic to L_τ, this will complete the proof.

Let B_i, $i = 1, \ldots, l$, be the possible standard tableaux obtained from C_σ by adding the element m to a single row (which must of course be the first row of its length), ordered with $B_1 \geqq B_2 \geqq \ldots$; for example if $\sigma = (3, 3, 2)$ and $m = 5$:

$$C_\sigma = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} \quad B_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array} \quad B_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 5 \\ \hline \end{array} \quad B_3 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Let σ^i be the shape of B_i. For $k = 1, \ldots, l$, let T_k be the space spanned by all tableaux of the form $(A | B_i)$ for $i = 1, \ldots, k$, with A arbitrary, so that $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_l$. By virtue of $(**)$ in Sect. 2, T_i has a basis consisting of the double standard tableaux that it contains. Thus we see that $T_k / T_{k-1} \cong L_{\sigma^k}$. Since $GL(n, F)$ is reductive, $T_l \cong \sum_k L_{\sigma^k}$, so

$$K_\sigma \bar{K}_1 \in T_l \subseteq I_\sigma.$$

5. The Arithmetic of Diagrams and the Arithmetic of Ideals

We will assume throughout this section that F is a field of characteristic 0. We will discuss primeness, radicals, and primary decompositions for G-invariant ideals.

In view of Corollary 4.2, it must be possible to treat all these notions purely in terms of diagrams. To this end, we will say that a set J of diagrams of $\leqq n$ columns is a D-ideal (for Diagrammatic ideal) if $\sigma \in J$ and $\tau \supseteq \sigma$ imply $\tau \in J$. Corollaries 4.2 and 3.4 yield:

Proposition 5.1. *There is a $1-1$ containment preserving correspondence between G-invariant ideals of R and D-ideals given by*

$$J \to I(J) = \sum_{\sigma \in J} M_\sigma \subset R$$

for a D-ideal J, and

$$I \to \{\sigma \mid K_\sigma \in I\} = \{\sigma \mid \bar{K}_\sigma \in I\} = \{\sigma \mid M_\sigma \subseteq I\}$$

for a G-invariant ideal I.

Mimicking the usual algebra, we will say that a G-ideal J is:

Prime if $\sigma \tau \in J$ and $\tau \notin J$ implies $\sigma \in J$;

Radical if $\sigma^p \in J$ implies $\sigma \in J$;

Primary if $\sigma \tau \in J$ and $\tau \notin J$ imply $\sigma^p \in J$ for some p.

Further, we write

$$\sqrt{J}=\{\sigma \mid \sigma^n \in J \text{ for some } n\}.$$

Clearly, the only radical D-ideals are the ideals

$$J_k=\{\sigma \mid \sigma_1 \geqq k\}, \qquad k=1,\ldots,n,$$

which are all prime.

The next result shows that the above definitions correspond to the usual ones:

Theorem 5.2. *A D-ideal J is prime, radical or primary if and only if the corresponding ideal $I(J)$ is. Further, $\sqrt{I(J)}=I(\sqrt{J})$. In particular, the only radical G-invariant ideals are the prime ideals I_k, $k=1,\ldots,n$.*

Proof. Since $K_\sigma K_\tau = K_{\sigma\tau}$ for any diagrams σ and τ, we see that if $I(J)$ is prime, radical, or primary, then J is; further, $\sqrt{I(J)} \supseteq I(\sqrt{J})$, and from the remarks above, $\sqrt{J}$ is always prime. It will thus suffice to show that if J is prime or primary, then $I(J)$ is.

Both these claims are immediate from the following: Let J be a primary D-ideal. By the remarks above, we have $\sqrt{J}=J_k$ for some k. We will show that the set of elements of R which are zero divisors modulo $I(J)$ is contained in I_k.

Suppose on the contrary that $f \in R - I_k$ is a zero-divisor modulo $I(J)$. Using the notation of the proof of Theorem 3.3 we have for some h

$$E_i^j(\lambda) f = \lambda^h f_h + (\text{terms of degree } < h \text{ in } \lambda)$$

for any $\lambda \in F$ and $i<j$. Since F is infinite, this implies that f_h is a zerodivisor $\operatorname{mod} I(J)$ too. Thus, as in the proof of Theorem 3.3, we may reduce to the case where f is an F-linear combination of anti-canonical tableaux, say $f=\sum a_i \bar{K}_{\sigma^{(i)}}$ with not all $\sigma^{(i)}$ lying in J_k.

Now suppose $fg \in I(J)$ with $g \notin I(J)$. Again for some h we have

$$\begin{aligned} E_i^j(\lambda) fg &= E_i^j(\lambda)(f)\, E_i^j(\lambda)(g) \\ &= f \cdot E_i^j(\lambda)(g) \\ &= \lambda^h \cdot f g_h + (\text{terms of degree } < h \text{ in } \lambda) \end{aligned}$$

for all λ and $i<j$. Since F is infinite, this implies $fg_h \in I(J)$. As above, we may assume that g is an F-linear combination of anti-canonical tableaux, $g=\sum_j b_j \bar{K}_{\tau^{(j)}}$, with $\tau^{(j)} \notin J$. Multiplying g by a power of $\bar{K}_{(k)}$ if necessary, and using the fact that $I(J)$ contains a power of $\bar{K}_{(k)}$, we may assume that $g\bar{K}_{(k)} \in I(J)$. Thus we may suppose that no $\sigma^{(i)}$ appearing in f lies in J_k.

Now suppose that $\sigma^{(1)}$ and $\tau^{(1)}$ are lexicographically latest among the $\sigma^{(i)}$ and $\tau^{(j)}$, respectively. Then $\sigma^{(1)}\tau^{(1)}$ is lexicographically latest among the $\sigma^{(i)}\tau^{(j)}$, so since

$$fg=\sum a_i b_j \bar{K}_{\sigma'(i)} \bar{K}_{\tau'(j)} = \sum a_i b_j \bar{K}_{\sigma(i)\tau(j)} \in I(J),$$

we must have $K_{\sigma^{(1)}\tau^{(1)}}$ in $I(J)$, or $\sigma^{(1)}\tau^{(1)} \in J$. Since J is primary and $\sigma^{(1)} \notin J_k$, we have $\tau^{(1)} \in J$, contradicting the construction.

Remark. The primeness of the ideals $I(J_k) = I_k$, which holds merely under the assumption that F is a domain, is of course well known; see for example [Hochster-Eagon]. An independent proof of this can easily be given by an induction similar to the one in the next section.

We will say that a diagram is *rectangular* of width k if all its rows have length k. The next result follows at once from the theorem and the definitions:

Corollary 5.3. *An ideal irredundantly written as* $\sum_{\sigma \in \Lambda} I_\sigma$ *is primary iff for some k we have* 1) *some* $\sigma \in \Lambda$ *is rectangular of width k and* 2) *no row of any* $\sigma \in \Lambda$ *has length* $<k$. *If* 1) *and* 2) *hold, then* $\sum_{\sigma \in \Lambda} I_\sigma$ *is* I_k*-primary.*

In particular, an ideal I_σ *is primary iff* σ *is rectangular.*

We now turn to the primary decomposition of G-invariant ideals, beginning with a special case.

Corollary 5.4. *Let* $n_1 > n_2 > \ldots > n_k$ *be the (distinct) lengths of rows in a diagram* σ, *and let* $\sigma^{(i)}$ *be the largest rectangular diagram with rows of length* n_i *such that* $\sigma^{(i)} \subseteq \sigma$. *Then*

$$I_\sigma = I_{\sigma^{(1)}} \cap \ldots \cap I_{\sigma^{(k)}}$$

is an irredudant primary decomposition of I_σ. *In particular, the associated primes of* I_σ *are* $I_{n_1}, \ldots, I_{n_k}$.

Proof. σ is the unique diagram minimal among the diagrams containing each $\sigma^{(i)}$; this would fail if some $\sigma^{(i)}$ were omitted. Thus the equality $I_\sigma = \cap I_{\sigma^{(i)}}$, as well as the irredudancy, follow from Proposition 5.1.

We now give an algorithm for forming a primary decomposition for any ideal $I = \sum_{\sigma \in \Lambda} I_\sigma$; the resulting decomposition may be redundant, but it is trivial to refine such a decomposition to an irredundant decomposition in any given case. (A general formula for an irredundant decomposition of products of determinantal ideals will be given later.)

Suppose p is the length of the shortest (nonzero) row in any $\sigma \in \Lambda$, and let $\tau \in \Lambda$ be an element in which a row of length p occurs, and such that τ has the minimal number s of rows among such elements. Write $\tau = (p)\tau'$ where (p) represents the diagram having one row, and that of length p. Let ρ be the rectangular diagram with s rows of length p, and let Λ' be $\Lambda - \{\tau\}$. We claim that

$$I = (I_\rho + \sum_{\sigma \in \Lambda'} I_\sigma) \cap (I_{\tau'} + \sum_{\sigma \in \Lambda'} I_\sigma).$$

The first ideal in the intersection on the right hand side is primary by Corollary 5.3, while we can suppose by induction that we are already in possession of a primary decomposition of the second ideal on the right side, so this will conclude the algorithm.

To establish the equality above, note first that the right hand side contains the left by Corollary 4.2. On the other hand if I_σ is contained in the right hand side, then either $\sigma \supseteq \lambda$ for some $\lambda \in \Lambda'$, in which case $I_\sigma \subseteq I$, or $\sigma \supseteq \rho$ and τ' in which case $\sigma \supseteq \tau$, and again $I_\sigma \subseteq I$.

6. Products of Determinantal Ideals

In this section, which is independent of Sects. 4 and 5, we will begin the study of determinantal ideals. Our central result identifies (in characteristic 0) the product $D_\sigma = I_{(\sigma_1)} I_{(\sigma_2)} \dots$ of determinantal ideals with the fundamental ideals A_σ (defined just before Corollary 2.3).

Theorem 6.1. *If F is a field of characteristic 0 and σ is a diagram, then $A_\sigma = D_\sigma$; equivalently, $\tau \geqq \sigma$ implies $D_\tau \subseteq D_\sigma$.*

For arbitrary F, this result fails; for instance, take $F = Z$, $n = m = 4$, $\tau = (3, 1) \geqq \sigma = (2, 2)$; then $D_\tau \nsubseteq D_\sigma$. In fact, in the free Z-module spanned by the forms of degree 4, modulo the standard monomials of shape (2, 2) and (4) the (nonstandard) monomials of shape (2, 2) generate a (free) Z-module of index 2 in the free Z-module generated by the standard monomials of shape (3, 1). More explicitly, $\left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ & & 4 & 4 & & \end{array}\right)$ is not in the square of the ideal of 2×2 minors. This may be shown by direct computation, using the straightening law.

However, a special case of Theorem 6.1, which we will use in the proof, is true in full generality:

Lemma 6.2. *If σ is a diagram of $\leqq 2$ rows, and $\tau \geqq \sigma$, then the doubly canonical tableau K_τ is in D_σ.*

We postpone the proof of the Lemma.

Proof of Theorem 6.1. By definition, A_σ is the F-linear span of the double tableaux of all shapes $\tau \geqq \sigma$. The double tableaux of shape τ are the generators of the ideal D_τ. Because of the form of the straightening formula, A_σ is an ideal, so $A_\sigma = \sum_{\tau \geqq \sigma} D_\tau$, and it suffices to prove that $A_\sigma \subseteq D_\sigma$ or, equivalently, that $D_\tau \subseteq D_\sigma$ for $\tau \geqq \sigma$.

This is obvious if σ has only one row.

If σ has just two rows, we use Corollary 3.4 to write $A_\sigma = \bigoplus_{\tau \geqq \sigma} M_\tau$, where each M_τ is irreducible and contains K_τ. It thus suffices to show that $K_\tau \in D_\sigma$ which is the conclusion of Lemma 6.2.

Suppose now that σ is arbitrary and $\tau \geqq \sigma$; we wish to show $D_\tau \subseteq D_\sigma$. We may assume that σ and τ are adjacent in the $\geqq$ order. By Proposition 1.2, σ_i and τ_i differ for at most two values of i. Thus we may write $\sigma = \kappa \sigma'$ and $\tau = \kappa \tau'$ with σ' a diagram of $\leqq 2$ rows. Since $D_\sigma = D_\kappa D_{\sigma'}$ and $D_\tau = D_\kappa D_{\tau'}$, the result follows from the case of 2 rows above.

It remains to prove the Lemma. For this purpose we will analyze the relations that hold over the integers between certain products of pairs of Plücker coordinates of a $k \times 2k$ matrix. Then we relate these to products of minors of an arbitrary matrix by the method already employed in Sect. 2.

We are grateful to S. Abeassis for pointing out various mistakes in the first version of this deduction.

Remark. In deriving relations with integral coefficients among the products of Plücker coordinates we may of course obtain a valid new relation by dividing an old one by an integer. (The result will certainly be valid over a field of characteristic 0, and thus holds in over the integers, so by specialization it holds over any ring.) However, the ideal generated by the "basic relations" 1) below does *not* contain the later relations! See [Abeassis].

We fix a new $k \times 2k$ matrix of indeterminates $Y=(Y_{ij})$. To each k element subset

$$A=\{i_1,\ldots,i_k\}\subset\{1,2,\ldots,2k\}$$

we associate a product of Plücker coordinates, as follows: Order the i_l so that $i_1<\ldots<i_k$, let $j_1<\ldots<j_k$ be the complementary subset, and set

$$\langle A\rangle=(-1)^{i_1+\cdots+i_k}[i_1,\ldots,i_k][j_1,\ldots,j_k].$$

We write $\langle i_1,\ldots,i_k\rangle$ in place of $\langle\{i_1,\ldots,i_k\}\rangle$. The symbol $\langle i_1,\ldots,i_k\rangle$ is symmetric in the i_j. We extend the definition to all sequences of k integers from $\{1,\ldots,2k\}$ by setting $\langle i_1,\ldots,i_k\rangle=0$ if $i_j=i_l$ for some $j\neq l$. If A' is the complement of A, then it is easy to see that $\langle A'\rangle=(-1)^k\langle A\rangle$.

We begin with a special case of the straightening law (the "exchange law"):

$$\begin{aligned}&[i_1\,i_2\ldots i_k][j_1\,j_2\ldots j_k]\\&\quad=\sum_{t=1}^{k}[j_t\,i_2\ldots i_k][j_1\ldots j_{t-1}\,i_1\,j_{t+1}\ldots j_k],\end{aligned}$$

where we assume $i_1<\ldots<i_k$, $j_1<\ldots<j_k$ and $i_1\ldots i_k\,j_1\ldots j_k$ is a permutation of $1,2,3,\ldots,2k$; this is the case $k=m$ of $(*)'$ in § 2.

We want to rewrite the terms in this identity using the elements $\langle j_t\,i_2\ldots i_k\rangle$. This will necessitate a computation of signs.

First of all

$$[i_1\,i_2\ldots i_k][j_1\,j_2\ldots j_k]=(-1)^{i_1+i_2+\cdots+i_k}\langle i_1\,i_2\ldots i_k\rangle$$

by definition.

Consider $[j_t\,i_2\ldots i_k][j_1\ldots j_{t-1}\,i_1\,j_{t+1}\ldots j_k]$. Suppose that s is such that $j_s<i_1<j_{s+1}$ and u is such that $i_u<j_t<i_{u+1}$ (s or u possibly zero). After reordering we have

$$\begin{aligned}&[j_t\,i_2\ldots i_k][j_1\ldots j_{t-1}\,i_1\,j_{t+1}\ldots j_k]\\&\quad=(-1)^{u+s+t}[i_2\ldots i_u\,j_t\,i_{u+1}\ldots i_k][j_1\ldots j_s\,i_1\,j_{s+1}\ldots\hat{j}_t\ldots j_k]\end{aligned}$$

(where $\hat{}$ means omitted).

Now i_1 is the smallest element in the complement of $j_1 j_2 \dots j_k$, so $s = i_1 - 1$; also, since the indices j are the complement of the i's, we have $u = j_t - t$ (there are exactly $j_t - t$ indices i less than j_t).

We get:

$$\begin{aligned}&[j_t i_2 \dots i_k][j_1 \dots j_{t-1} i_1 j_{t+1} \dots j_k]\\ &\quad = -(-1)^{j_t + i_1}[i_2 \dots i_u j_t i_{u+1} \dots i_k][j_1 \dots j_s i_1 j_{s+1} \dots j_t \dots j_k]\\ &\quad = -(-1)^{j_t + i_1 + i_2 + i_3 + \dots + i_k + j_t}\langle i_2 \dots i_u j_t i_{u+1} \dots i_k\rangle\\ &\quad = -(-1)^{i_1 + i_2 + \dots + i_k}\langle i_2 \dots i_u j_t i_{u+1} \dots i_k\rangle.\end{aligned}$$

Thus, taking into account our convention on repeated indices, the exchange relation may be written

$$\sum_{j=1}^{2k} \langle j, i_2, \dots, i_k\rangle = 0. \tag{1}$$

Using induction on an integer s, we can at once generalize this to

$$\sum_{j_1 = j_2 = \dots = j_s = 1}^{2k} \langle j_1, \dots, j_s, i_{s+1}, \dots, i_k\rangle = 0, \tag{2}$$

the sum being taken over sequences j with $j_1 \leqq j_2 \leqq \dots \leqq j_s$.

Proof. Writing $R_{i_{s+1}, \dots, i_k}$ for the sum in question, we have $\sum_{j=1}^{k} R_{j, i_{s+1}, \dots, i_k} = s R_{i_{s+1}, \dots, i_k}$, so (2) follows by induction, using the remark on division by integers. //

We can now deduce the relation needed for Lemma 6.2, which is:

$$\begin{aligned}&\sum_{\{j_1 \dots j_s\} \cap \{i_1 \dots i_s\} = \emptyset} \langle j_1 j_2 \dots j_s i_{s+1} \dots i_k\rangle\\ &\quad = (-1)^s \langle i_1 i_2 \dots i_k\rangle.\end{aligned} \tag{3}$$

The proof is again by induction on s:

For $s = 1$, (3) becomes (1):

$$-\langle i_1 \dots i_k\rangle = \sum_{j \neq i_1} \langle j i_2 \dots i_k\rangle.$$

Look at the relation $R_{i_{s+1} \dots i_k} = 0$. We can split its terms in two parts; those for which $j_1 \dots j_s$ are distinct from i_s and the others where i_s appears:

$$\begin{aligned}0 = R_{i_{s+1} \dots i_k} &= \sum_{j_1 j_2 \dots j_s \neq i_s} \langle j_1 \dots j_s i_{s+1} \dots i_k\rangle\\ &\quad + \sum \langle j_1 \dots j_{s-1} i_s i_{s+1} \dots i_k\rangle.\end{aligned}$$

The second term is just $R_{i_s i_{s+1} \dots i_k}$, which is 0, for $s \geqq 2$, and hence for $s \geqq 2$, $\sum_{j_1 \dots j_s \neq i_s} \langle j_1 \dots j_{s+1} \dots i_k\rangle = 0$. Next we split the remaining terms in the ones distinct from i_{s-1} and the ones containing i_{s-1}. If $s \geqq 3$, the sum

$\sum_{j_1,\ldots,j_{s-1}\neq i_s} \langle j_1 \ldots j_{s-1}\, i_{s-1}\, i_{s+1} \ldots i_k\rangle$ is of the type just proved zero, so we may continue. We finally get

$$0=\sum_{j_1 j_2 \ldots j_s \neq i_1 \ldots i_s} \langle j_1 \ldots j_s\, i_{s+1} \ldots i_k\rangle$$
$$+\sum_{j_1 \ldots j_{s-1} \neq i_2 \ldots i_s} \langle j_1 \ldots j_{s-1}\, i_1\, i_{s+1} \ldots i_k\rangle.$$

The second term is by induction $(-1)^{s-1}\langle i_1\, i_2 \ldots i_k\rangle$, hence the claim follows.

Proof of Lemma 6.2. We begin by specializing Y to the $k\times 2k$ matrix $\bar{Y}=(X\,|\,\bar{X})$, where X is a generic $k\times k$ matrix $X=(X_{ij})$ and

$$\bar{X}=\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1h} & & 0 & \\ x_{21} & x_{22} & \cdots & x_{2h} & & & \\ \vdots & \vdots & & \vdots & 1 & & 0 \\ & & & & & 1 & \\ & & & & & & \ddots \\ x_{k1} & x_{k2} & \cdots & x_{kh} & 0 & & 1 \end{pmatrix} \begin{matrix} \Big\}\, h \\ \\ \Big\}\, k-h \end{matrix}$$
$$\underbrace{\qquad\qquad}_{k-h}$$

that is, the first h columns of $\bar{X}$ are equal to the first h columns of X, and the last $k-h$ columns of $\bar{X}$ are equal to the last $k-h$ columns of the identity matrix. Some of the maximal minors of $\bar{Y}$ are minors of X, and some vanish because of the repetitions of columns.

We apply (3) above to the product

$$\langle k-l+1, k-l+2, \ldots, k, 1, \ldots, k-l\rangle = \pm[1,2,\ldots,k]\,[k+1,\ldots,2k],$$

assuming $k-l\geqq h$. We see that $[1,2\ldots k]\,[k+1\ldots 2k]$ is equal to a sum of products of the form

$$\pm[1,2\ldots k-l\, u_1\, u_2 \ldots u_l]\,[k-l+1 \ldots k\, u_{l+1} \ldots u_k]$$

where $u_1, u_2, \ldots, u_k$ is a permutation of $k+1, k+2, \ldots, 2k$.

The only nonzero terms are those in which all the u_i's, $i=1,2,\ldots,l$, are greater than $k+h$, since the others are determinants of matrices with two columns repeated. After the specialization we have:

1) $[1,2\ldots k]\,[k+1\ldots 2k] = \pm(k\ldots 2\,1|1\,2\ldots k)\,(h\,h-1\ldots 2\,1|1\,2\ldots h)$
$$= \pm K_{(k,h)}.$$

2) The terms $[1\,2\ldots k-l\, u_1 \ldots u_l]\,[k-l+1\ldots k\, u_{l+1}\ldots u_k]$ are products of $k-l$ minors of X times $h+l$ minors.

Thus we get the conclusion of the Lemma for the case $\tau=(k,h)$, $\sigma=(k-l, h+l)$, with $k-l\geqq h$; that is, the case $\tau\geqq\sigma$ with $\tau_1+\tau_2=\sigma_1+\sigma_2$.

If now τ is any partition $\geqq\sigma$, then there is a partition $\sigma'=(\sigma'_1,\sigma'_2)$ such that $\tau\geqq\sigma'\supseteq\sigma$ and $\tau_1+\tau_2=\sigma'_1+\sigma'_2$. The Laplace expansion of a minor in terms of products of smaller minors shows at once that $\sigma'\supseteq\sigma$ implies $D_{\sigma'}\subseteq D_\sigma$, so we have

$$K_\tau \in D_{\sigma'} \subseteq D_\sigma,$$

as required.

7. Symbolic Powers and Order of Vanishing on Determinantal Varieties

The notions in the title of this section are connected by the fact that if P is a prime ideal in a polynomial ring over, say a field, then the P-primary component $P^{(n)}$ of P^n is the set of functions vanishing to order $\geqq n$ at every closed point of the variety defined by P. The ideal $P^{(n)}$, called the n^{th} symbolic power of P, may be defined by

$$P^{(n)} = \{f \mid \text{there exists } s \notin P \text{ with } sf \in P^n\}.$$

The statement of the geometric significance of $P^{(n)}$ above is due to Zariski (partly in [Zariski]); see [Eisenbud-Hochster] for further details and a modern treatment.

In this section we will determine the ideals $I_k^{(p)}$ and the irredundant primary decomposition, which can be made in terms of them, of the ideals A_σ. We will assume only that F is a domain. In characteristic 0 we thus get the irredundant primary decomposition of any product of determinantal ideals D_σ.

We begin by determining the ideals $I_k^{(p)}$:

Theorem 7.1. *If F is an integral domain, then*

1) *$I_k^{(p)}$ is the linear span of all the double tableaux of shape σ with $\gamma_k(\sigma) \geqq p$; it thus has an F-free basis consisting of the double standard tableaux of those shapes.*

2) $$I_k^{(p)} = \sum_{\gamma_k(\sigma) \geqq p} D_\sigma = \sum I_k^{p_1} I_{k+1}^{p_2} \dots I_{k+l-1}^{p_l},$$

where the second sum extends over all the sequences of nonnegative integers $p_1, \dots, p_l$ with $k+l-1 \leqq n$ and $p_1 + 2p_2 + \dots + lp_l \geqq p$.

Before giving the proof of the theorem, it may be illuminating to prove the first statement of 1) in the case where F is a field of charactersitic 0, so that the order of vanishing of f on a variety V is the largest l such that f and all its derivatives of orders $< l$ vanish on V. In this case it is enough to show that a standard tableau T of shape $\sigma = (\sigma_1, \sigma_2 \dots)$ vanishes on the variety V_{k-1} associated to I_k exactly to order $\gamma_k(\sigma)$. Now T is a product of minors of order $\sigma_1, \sigma_2, \dots$, and the order of vanishing of T is the sum of the order of vanishing of these minors. Since $\gamma_k(\sigma) = \sum_{\sigma_i \geqq k} \sigma_i - k + 1$, it suffices to show that a minor of order h vanishes on V_{k-1} to order precisely $h-k+1$. However, the partial derivatives of order t of the minor with respect to some of the X_{ij} are either equal to 0 or to a scalar multiple of a subminor of order $h-t$, and every subminor of order $h-t$ appears; thus the derivatives of order $h-k+1$ are contained in I_k (vanish on V_{k-1}) while those of order $h-k$ do not.

Proof of Theorem 7.1. For convenience, set

$$J_{k,p}=\sum_{\gamma_k(\sigma)\geqq p} D_\sigma.$$

We will prove by induction on k that $J_{k,p}=I_k^{(p)}$, thus establishing part 2. The ideal $J_{k,p}$ is clearly the span of the double tableaux of shape σ with $\gamma_k(\sigma)\geqq p$; because of the form of the straightening law, $J_{k,p}$ admits as a basis the double standard tableaux of these shapes. Thus part 1) of the theorem will follow from part 2).

If $k=1$, then the powers of I_k are easily seen to be primary, that is, $I_k^{(p)}=I_k^p$. Since I_1^p is the ideal of all polynomials of degree $\geqq p$ in the X_{ij}, we have $I_1^p=J_{1,p}$.

Now suppose that $k\geqq 2$, and that the theorem is satisfied for smaller k and every domain F. With this hypothesis we claim that X_{nm} is a nonzerodivisor modulo $J_{k,p}$. For, if T_i are double standard tableaux of shapes $\sigma^{(i)}$, then $T_i X_{nm}$ are again double standard and have $\sigma^{(i)}\cdot(1)$. But

$$\gamma_k(\sigma^{(i)}\cdot(1))=\gamma_k(\sigma^{(i)})+\gamma_k((1))=\gamma_k(\sigma^{(i)}) \qquad (k\geqq 2),$$

so

$$(\textstyle\sum \lambda_i T_i)\, X_{nm}\in J_{k,p} \qquad \text{iff } \textstyle\sum \lambda_i T_i\in J_{k,p},$$

as required.

Now $X_{nm}\notin I_k$, since I_k is generated by forms of degree k, so X_{nm} is automatically a nonzerodivisor modulo $I_k^{(p)}$. It thus suffices to show that $I_k^{(p)}=J_{k,p}$ after inverting X_{nm}.

Over the ring $R'=R[X_{nm}^{-1}]$ the matrix X can be transformed by elementary row and column operations, and multiplication by X_{nm}^{-1}, to a matrix of the form

$$\begin{pmatrix} X'_{11} & \dots & X'_{1,m-1} & 0 \\ X'_{21} & \dots & X'_{2,m-1} & 0 \\ \vdots & & \vdots & \vdots \\ X'_{n-1,1} & \dots & X'_{n-1,m-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

where $X'_{ij}=X_{ij}-X_{nj}X_{im}X_{nm}^{-1}$, for $1\leqq i\leqq n-1$ and $1\leqq j\leqq m-1$. Writing X' for the $n-1\times m-1$ matrix (X'_{ij}), the matrix above is $X'\oplus 1$, and the ideal $I_k R'$ is the ideal of R' generated by the k-order minors of $X'\oplus 1$, or, equivalently, by the $(k-1)$ order minors of X', which we write $I_{k-1}(X')$. Similarly, we may, with an obvious notation, write

$$J_{k,p}(X)\,R'=\sum_{\gamma_{k-1}(\sigma)\geqq p} D_\sigma(X').$$

We now wish to use the induction hypothesis to complete the proof.

For this, it is enough to show that R' is a polynomial ring on the variables X'_{ij} over a suitable integral domain. We claim that

$$R'\cong F'[\tilde{X}'_{ij}]$$

where

$$F' = F[X_{n1}, \ldots, X_{nm}, X_{1m}, \ldots, X_{n-1,m}, X_{nm}^{-1}]$$

and the $\tilde{X}_{ij}$ are indeterminates, which go to X'_{ij} under the isomorphism.

To prove this note that there is an evident epimorphism $F'[\tilde{X}'_{ij}] \to F[X_{ij}][X_{nm}^{-1}] = R'$. Tensoring both rings, over the common subring F', with the quotient field of F', we get an epimorphism between equi-dimensional integral domains of the same krull dimension: such a map must be an isomorphism. Alternatively, one can write down the inverse map using the equations

$$X'_{ij} = X_{ij} - X_{nj} X_{im} X_{nm}^{-1}.$$

Thus, by induction, $I_{k-1}^{(p)}(X') = J_{k-1,p}(X')$, completing the proof.

We can now give the primary decomposition for the ideals A_σ. Recall that we write $\check{\sigma}_k$ for the length of the k^{th} column of a diagram $\check{\sigma}$, and that all diagrams are assumed to have $\leqq n$ columns, where n is the number of rows ($\leqq$ the number of columns) of the matrix (X_{ij}).

Theorem 7.2. *Let σ be a diagram, and suppose F is a domain. A primary decomposition for $A_\sigma \subset R$ is given by*

$$A_\sigma = \bigcap_{1 \leqq k \leqq n} I_k^{(\gamma_k(\sigma))}.$$

This decomposition is irredundant if the intersection is restricted to those k for which $\check{\sigma}_k \neq 0$ and $\gamma_{k+1}(\sigma) \leqq (n-k)(\check{\sigma}_k - 1)$.

If F is a field of characteristic 0, then $A_\sigma = D_\sigma$. As an interesting special case we obtain

Corollary 7.3. *If F is a field of characteristic 0, then the irredundant primary decomposition of a power of a determinantal ideal is given by*

$$I_j^p = I_j^{(p)} \cap I_{j-1}^{(2p)} \cap \ldots \cap I_l^{((j-l+1)p)},$$

where $l = \max(1, n - p(n-j))$.

The Corollary follows from the theorem by straightforward arithmetic, since $I_j^p = D_\sigma$ with $\sigma = (j, j, \ldots, j)$ (p rows), $\check{\sigma}_k = p$ for $k = 1, \ldots, j$, and $\gamma_{k+1}(\sigma) = p(j-k)$ for $k \leqq j$.

Proof of Theorem 7.2. From Theorem 7.1, we see that an ideal of the form $\bigcap_{k \in K} I_k^{(\gamma_k(\sigma))}$ has a basis consisting of double standard tableaux of shape τ with $\gamma_k(\tau) \geqq \gamma_k(\sigma)$ for all $k \in K$, while A_σ has as basis the set of double standard tableaux of shape τ with $\tau \geqq \sigma$, that is, $\gamma_k(\tau) \geqq \gamma_k(\sigma)$ for all $k = 1, \ldots, n$. Thus $A_\sigma = \bigcap_{1 \leqq k \leqq n} I_k^{(\gamma_k(\sigma))}$, and the second statement of Theorem 7.2 is implied by Proposition 1.4.

8. Integral Closures of G-Invariant Ideals

Integral closures of ideals are important for two reasons. First, in algebraic geometry, taking an integral closure corresponds under certain conditions to completing a linear system. Second, in an analytic setting (e.g., $F=\mathbb{C}$ or $\mathbb{R}$), the integral closure of an ideal generated by functions $f_1, \ldots, f_k$ is the set of functions g such that, for some constant C,

$$|g(x)| \leqq C \sum_i |f_i(x)| \qquad \text{for all } x.$$

(See [Zariski-Samuel, Appendix 4 to Vol. II], and [Lejeune-Teissier] for details.) In this section we will (following a suggestion of Kempf) compute the integral closures of G-invariant ideals. We will assume throughout that F is a field of characteristic 0.

The key result is the following:

Theorem 8.1. *The integral closure of I_σ is D_σ.*

Proof. Since we are in characteristic 0, $D_\sigma = A_\sigma$, so by Theorem 7.2 we have

$$D_\sigma = \bigcap_{1 \leqq k \leqq n} I_k^{(\gamma_k(\sigma))}.$$

But symbolic powers of prime ideals in a regular ring are integrally closed (it is enough to show this for the maximal ideal; and in this case the fact that the associated graded ring is a polynomial ring makes the result easy), as are intersections of integrally closed ideals. Thus D_σ is integrally closed.

It remains to show that D_σ is contained in the integral closure of I_σ. Since the integral closure of a G-invariant ideal is G-invariant, it suffices to show that the doubly canonical tableaux K_τ is integral over I_σ for all $\tau \geqq \sigma$; in fact, by Lemma 3.4, it sufficies to show that the integral closure of I_σ contains some double tableau of shape τ.

Since integrality is transitive, we may assume that τ is adjacent to σ. If $\tau \supseteq \sigma$, then by Theorem 4.1 $I_\sigma \supseteq I_\tau$. Using Proposition 1.2, we may thus assume that

$$\tau = \sigma - \{(j, h)\} \cup \{(i, k+1)\} \qquad \text{for some } j > i, \text{ and } h \leqq k.$$

We will show in this case that there is a double tableau of shape $\tau \cdot \tau$ contained in I_σ^2, thus completing the proof (again use Lemma 3.4).

Consider the semicanonical tableaux $\tilde{K}_\sigma$ (respectively $\tilde{\tilde{K}}_\sigma$) obtained by substituting $k+1$ for h whereever it occurs in the left (respectively right) part of K_σ. We may write $\tilde{K}_\sigma \cdot \tilde{K}_\sigma = \bar{K} \cdot a$, where a is the double tableau of 4 rows of shape (k, k, h, h), coming from the i^{th} and j^{th} rows of $\tilde{K}_\sigma$ and $\tilde{\tilde{K}}_\sigma$:

$$a = (A \mid B)$$

$$= \left(\begin{array}{cccccc|cccccc} k & k-1 & \ldots & h & h-1 & \ldots\ 1 & 1 & \ldots & h-1 & h+1 & \ldots\ \ k & k+1 \\ k+1 & k & \ldots & h+1 & h-1 & \ldots\ 1 & 1 & \ldots & h-1 & h & \ldots\ k-1 & k \\ & & & k+1 & h-1 & \ldots\ 1 & 1 & \ldots & h-1 & h & & \\ & & & h & h-1 & \ldots\ 1 & 1 & \ldots & h-1 & k+1 & & \end{array}\right).$$

Writing c for the canonical double tableaux of shape $(k+1, k+1, h-1, h-1)$, we will show that $\bar{K}\cdot c$, which has shape $\tau\cdot\tau$, is in I_σ^2 by applying the quadratic relations of the straightening law to $\tilde{K}_\sigma \tilde{\tilde{K}}_\sigma = \bar{K} a \in I_\sigma^2$, and to some related tableaux.

We first define a family $a_{r,t}$ of double tableaux with the property that $\bar{K} a_{r,t} \in I_\sigma^2$. We begin by setting $a = a_{h,k+1}$.

For $h+1 \leqq r \leqq k+1$ we set:

$$a_{r,k+1} = \left(\begin{array}{cccccccccc|c} k & \dots & r & r-1 & r-2 & \dots & h & h-1 & \dots & 1 & \\ k+1 & \dots & r+1 & h & r-1 & \dots & h+1 & h-1 & \dots & 1 & B \\ & & & & & & k+1 & h-1 & \dots & 1 & \\ & & & & & & r & h-1 & \dots & 1 & \end{array}\right),$$

the result of exchanging the r in the second row with the h in the fourth row of A in $(A|B)$.

For $h \leqq t \leqq k$, we set:

$$a_{h,t} = \left(\begin{array}{cccccccccc|c} k & \dots & t+1 & k+1 & t-1 & \dots & h & h-1 & \dots & 1 & \\ k+1 & \dots & t+2 & t+1 & t & \dots & h+1 & h-1 & \dots & 1 & B \\ & & & & & & t & h-1 & \dots & 1 & \\ & & & & & & h & h-1 & \dots & 1 & \end{array}\right),$$

the result of exchanging t in the first row with $k+1$ in the third row of A in $(A|B)$.

For $h+1 \leqq r \leqq k+1$ and $h \leqq t \leqq k$ we set $a_{r,t}$ equal to the double tableau made from the first and third rows of $a_{h,t}$ and the second and fourth rows of $a_{r,k+1}$.

For all $h \leqq r$, $t \leqq k+1$, the double tableau $\bar{K} a_{r,t}$ has the same right hand tableau as $\tilde{K}_\sigma \tilde{\tilde{K}}_\sigma$, and thus $\bar{K} a_{r,t}$ is the product of a right semicanonical tableau of shape σ with a double tableau obtained from a right semicanonical tableau of shape σ by transposing the h^{th} and $k+1^{\text{st}}$ columns. Thus by Lemma 3.4, $\bar{K} a_{rt} \in I_\sigma^2$.

We now apply the straightening law in the form of the quadratic relation (**) of Sect. 2 to $a_{r,k+1}$, using the row indices in the first and third rows.

Since the first row already contains $1, \dots, h-1$, only $k+1$ in the third row and $h, \dots, k$ in the first row will be involved, and the relation takes the form

$$a_{r,k+1} = \sum_{h \leqq t \leqq k} \pm a_{r,t} \pm b_r,$$

where b_r is defined for $h \leqq r \leqq k+1$ by:

$$b_h = (A'|B')$$

$$= \left(\begin{array}{cccccccc|cccccc} k+1 & k & k-1 & \dots & h & h-1 & \dots & 1 & 1 & \dots & h-1 & h & \dots & k \;\; k+1 \\ & k+1 & k & \dots & h+1 & h-1 & \dots & 1 & 1 & \dots & h-1 & h & \dots & k \\ & & & & h & h-1 & \dots & 1 & 1 & \dots & h-1 & k+1 & & \\ & & & & & h-1 & \dots & 1 & 1 & \dots & h-1 & & & \end{array}\right)$$

the result of moving $k+1$ from the third to the first row in A and moving h from the third to the first row in B, and rearranging the rows; while for

$h+1 \leqq r \leqq k+1$ we have

$$b_r = \left(\begin{matrix} k+1 & k & k-1 & \dots & r & r-1 & r-2 & \dots & h & h-1 & \dots & 1 \\ & k+1 & k & \dots & r+1 & h & r-1 & \dots & h+1 & h-1 & \dots & 1 \\ & & & & & & & & r & h-1 & \dots & 1 \\ & & & & & & & & & h-1 & \dots & 1 \end{matrix} \middle| B' \right),$$

the result of excahnging r in the second row with h in the third row of A' in $(A'|B')$.

Since $\bar{K} a_{r,t} \in I_\sigma^2$, we will have $\bar{K} b_r \in I_\sigma^2$, too. If we now apply (**) of Sect. 2 to b_h, using the row indices in the second and third rows, we obtain

$$b_h = \sum_{h+1 \leqq r \leqq k+1} \pm b_r \pm c,$$

so it follows that $\bar{K} c \in I_\sigma^2$, as desired. //

We can now compute the integral closure of any G-invariant ideal. As in the case of ideals generated by ordinary monomials in the variables, integral closure turns out to be related to convexity. To discuss this we embed the partially ordered set of diagrams with $\leqq n$ columns into $\mathbb{R}^n$, which we partially order by the definition $(a_1, \dots, a_n) \leqq (b_1, \dots, b_n)$ iff $a_i \leqq b_i$ for each i, by the map $\sigma \mapsto \gamma(\sigma) = (\gamma_1(\sigma), \dots, \gamma_n(\sigma))$.

As in Sect. 5, we may write any invariant ideal in the form $I(J) = \sum_{\sigma \in J} I_\sigma$ for some D-ideal J. We define the *integral closure* $\bar{J}$ of J by setting

$$\bar{J} = \{\sigma \mid \gamma(\sigma) \geqq x \text{ for some } x \text{ in the convex hull of } \gamma(J)\}.$$

Theorem 8.2. *If J is a D-ideal, then $I(\bar{J})$ is the integral closure of $I(J)$.*

Proof. By Theorem 8.1, the integral closure $\overline{I(J)}$ of $I(J)$ contains D_σ for each $\sigma \in J$. Thus it suffices to show that $K_\tau \in \overline{\sum_{\sigma \in J} D_\sigma}$ iff $\tau \in \bar{J}$.

On the one hand, if $\tau \in \bar{J}$, then for some $\tau_i \in J$, we have $\gamma(\tau) \geqq \sum a_i \gamma(\sigma_i)$ with $\sum a_i = 1$, and $a_i \geqq 0$. We may assume the a_i are rational, and write $a_i = b_i/c$ with b_i, c positive integers and $\sum b_i = c$. Thus

$$\gamma(\tau^c) = c\,\gamma(\tau) \geqq \sum b_i \gamma(\sigma_i) = \gamma(\Pi\, \sigma_i^{b_i}),$$

whence $K_{\tau^c} \in \Pi\, D_{\sigma_i}^{b_i} \subset (\sum_{\sigma \in J} D_\sigma)^c$, and $K_\tau \in \overline{\sum_{\sigma \in J} D_\sigma}$.

Next suppose K_τ is integral over $D = \sum_{\sigma \in J} D_\sigma$, so that for some c, $K_\tau^c \in K_\tau^{c-1} D + K_\tau^{c-2} D^2 + \dots + D^c$. The right hand side is by Theorem 6.1 spanned by double standard tableaux of shape $\geqq \sigma_1 \sigma_2 \dots \sigma_k \tau^{c-k}$ for some $\sigma_1, \dots, \sigma_k \in J$ with $c \geqq k \geqq 1$. Thus we can write $\tau^c \geqq \sigma_1 \sigma_2 \dots \sigma_k \tau^{c-k}$, whence $\tau^k \geqq \sigma_1 \dots \sigma_k$. Applying γ, we see that $\gamma(\tau) \geqq \frac{1}{k} \sum_1^k \gamma(\sigma_i)$, so $\tau \in \bar{J}$ as desired. //

We close with two examples:

1) Not every sum of determinantal ideals is integrally closed. Setting $\sigma_1 = (4, 4)$, $\sigma_2 = (3, 3, 3, 3)$, $\tau = (4, 3, 3)$, we have $\tau^2 = \sigma_1 \sigma_2$, so K_τ is integral over $D_{\sigma_1} + D_{\sigma_2}$, but $\tau \ngeq \sigma_1, \sigma_2$, so $K_\tau \notin D_{\sigma_1} + D_{\sigma_2}$.

2) *Even if an ideal is integrally closed, its square need not be: In* $\mathbb{R}^6$, *consider the vectors*

$$p_1 = (1, 1, 1, 0, 0, 0),$$

$$p_2 = (1, 0, 0, 1, 1, 0),$$

$$p_3 = (0, 1, 0, 1, 0, 1),$$

$$p_3 = (0, 0, 1, 0, 1, 1),$$

$$q = (30, 20, 12, 6, 2, 0).$$

For each i, we may write $q + p_i = \gamma(\sigma_i)$ for suitable σ_i, and with $I = D_{\sigma_1} + D_{\sigma_2} + D_{\sigma_3} + D_{\sigma_4}$, one checks by Theorem 8.2 that I is integrally closed. On the other hand, the diagram τ for which $\gamma(\tau) = 2q + (1, 1, 1, 1, 1, 1)$ satisfies $\gamma(\tau^2) = \sum_1^4 (q + p_i)$, so $K_\tau^2 \in I^4$, but $K_\tau \notin I^2$.

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