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ENRICHED FREE RESOLUTIONS AND CHANGE OF RINGS

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Let A be a commutative local ring and let $B = A/I$ be a factor ring. In [B-E] it is shown that any A-free resolution of B possesses the structure of a homotopy-associative, commutative differential graded algebra. This algebra structure, which generalizes the exterior algebra structure of the Koszul complex, can be quite useful in the analysis of free resolutions (The results of [B-E] are one exemple) The idea of its construction is as follows : If

$$\mathbb{F} : \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow A$$

is an A-free resolution of B, then $S_2(\mathbb{F})$, the symmetric square of $\mathbb{F}$, inherits the structure of a complex from $\mathbb{F} \otimes \mathbb{F}$. The first few terms of $S_2(\mathbb{F})$ are as follows :

$$\begin{tikzcd}
\cdots \arrow[r] & F_4 \otimes A \arrow[r] & F_3 \otimes A \arrow[r] & F_2 \otimes A \arrow[r] & F_1 \otimes A \arrow[r] & S_2(A) \\
& \oplus & \oplus & \oplus & & \Vert \\
\cdots \arrow[r] \arrow[uur] & F_3 \otimes F_1 \arrow[r] \arrow[uur] & F_2 \otimes F_1 \arrow[r] \arrow[uur] & \wedge^2 F_1 \arrow[uur] & & A \\
& \oplus & & & & \\
\cdots \arrow[r] \arrow[uur] & S_2(F_2) \arrow[uur] & & & &
\end{tikzcd}$$

The natural isomorphism $F_i \otimes A \longrightarrow F_i$ extends, uniquely up to homotopy, to a map of complexes $S_2(\mathbb{F}) \longrightarrow \mathbb{F}$, and this map, combined with the natural map :

$$\mathbb{F} \otimes \mathbb{F} \longrightarrow S_2(F)$$

acts as the multiplication map for the algebra structure on $\mathbb{F}$.

What about the resolution of a module ? If M is an A-module annihilated by I-that is, if M is a $B = A/I$ - module - and if $\mathbb{G}$ is a free resolution of M, then the natural map $B \otimes_A M \longrightarrow M$ extends, uniquely up to homotopy, to a map :

$$\mathbb{F} \otimes_A \mathbb{G} \longrightarrow \mathbb{G}$$

which makes $\mathbb{G}$ into a homotopy associative, differential graded $\mathbb{F}$-module.

In [B-E] we conjectured that the comparaison map $S_2(\mathbb{F}) \longrightarrow \mathbb{F}$, above, could be chosen in such a way that the algebra structure on $\mathbb{F}$ is associative (not just up to homotopy). It seems reasonable to extend this and to conjecture that, with a suitable associative algebra structure on $\mathbb{F}$, the resolution $\mathbb{G}$ can be made into an associative $\mathbb{F}$-module.

Considere now the case in which I is generated by a regular sequence, so that $\mathbb{F}$ is the Koszul complex, which is naturally an (associative) differential graded commutative algebra (in fact the underlying algebra is an exterior algebra). In this case it is easy to say what an $\mathbb{F}$-module structure on $\mathbb{G}$ looks like (see Proposition 1), and in certain cases-for example when the module M resolved by $\mathbb{G}$ is just the residue class field-it is clear that one exists. Moreover, if one examines the construction, due to Tate, of a B-free resolution of the residue class field from an A-free resolution, one sees that the data required by Tate to make the construction is equivalent to the data involved in constructing an $\mathbb{F}$-module structure on $\mathbb{G}$.

One problem in extending Tate's idea to resolution $\mathbb{G}$ of an arbitrary B-module M is the possible non-associativity of $\mathbb{G}$. In this paper we will show (Theorem 2) how an analogue of Tate's construction can be carried through, for arbitrary $\mathbb{G}$, (but always under the assumption that $\mathbb{F}$ is a Koszul complex) by considering not only the algebra structure on $\mathbb{G}$ but also a sort of "higher homotopy associativity" that $\mathbb{G}$ satisfies even if it is not associative (Theorem 1).

It would be interesting to know the right generalization of this "higher homotopy associativity" for more general (perhaps non-associative) $\mathbb{F}$, and to know how, in general, to derive resolutions over B from resolutions over A (The evidence in [Lev] and [Ar] seems to point to the introduction of matric Massey products for this purpose).

From now on, suppose that $I = (x_1,\dots,x_n)$ where $x_1,\dots,x_n$ form an A-sequence. The minimal free resolution of $B = A/I$ is then the Koszul complex :

$$\Lambda : 0 \longrightarrow \Lambda^n A^n \longrightarrow \cdots \longrightarrow \Lambda^3 A^n \xrightarrow{\delta} \Lambda^2 A^n \xrightarrow{\delta} A^n \xrightarrow{\delta} A.$$

Choose a basis $\varepsilon_1,\dots,\varepsilon_n \in A^n$ so that $\delta : \varepsilon_i \longmapsto x_i \in A$. The following is immediate from the definition of a differential graded module :

<u>Proposition 1</u> <u>If</u>

$$\mathbb{G} : \cdots \xrightarrow{\partial} G_k \xrightarrow{\partial} G_{k-1} \xrightarrow{\partial} \cdots$$

<u>is a differential graded A-module, then an associative differential graded Λ-module structure on $\mathbb{G}$ is equivalent to a set of n maps of graded A-modules of degree $+1$:</u>

$$s_1,\dots,s_n : \mathbb{G} \longrightarrow \mathbb{G}$$

<u>satisfying :</u>

(1) $s_i\partial + \partial s_i = x_i \cdot 1$

(2) $s_i s_j = - s_j s_i$, $s_i^2 = 0$.

Now if $\mathbb{G}$ is an A-free resolution of a B-module M, then multiplication by x_i is 0 in M, so $x_i.1 : \mathbb{G} \longrightarrow \mathbb{G}$ is homotopic to 0 ; to say that $s_i : \mathbb{G} \longrightarrow \mathbb{G}$ is a homotopy for $x_i.1$ is exactly condition (1) of the proposition, so maps satisfying condition (1) do indeed exist (Alternatively, these maps could be constructed by choosing a map of complexes $\Lambda \otimes \mathbb{G} \longrightarrow \mathbb{G}$ and regarding the induced map $A^n \otimes \mathbb{G} \longrightarrow \mathbb{G}$ as giving n maps $\mathbb{G} \longrightarrow \mathbb{G}$ of degree 1). Unfortuneatly there is no reason why an arbitrarily choosen set of maps s_i should satisfy (2).

On the other hand, the homotopy-uniqueness of the s_i implies that $s_i s_j - s_j s_i$ is at least homotopic to 0. Theorem 1 extends this remarks and tell us what we may expect from a random choice of homotopies.

We first introduce some multi-index conventions. A multi-index (of length n) is a sequence :

$$\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$$

where each α_i is an integer $\geqslant 0$. We write :

$$0 = \langle 0, \ldots, 0 \rangle .$$

The order of a multi-index is :

$$|\alpha| = \sum_{i=1}^{n} \alpha_i .$$

The sum $\alpha + \beta$ is defined as $\langle \alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n \rangle$.

<u>Theorem 1</u> : <u>Let</u> A <u>be a ring and let</u> M <u>be an A-module which is annhilated by elements</u> $x_1, \ldots, x_n \in A$. <u>Suppose the ideal</u> $(x_1, \ldots, x_n)$ <u>contains a non zero divisor. If</u> $\mathbb{G}$ <u>is a free resolution of</u> M, <u>then there are endomorphisms</u> s_α <u>of degree</u> $2|\alpha| - 1$ <u>of</u> $\mathbb{G}$ <u>as a graded module, for each multi-index</u> α , <u>satisfying</u> :

i) s_0 <u>is the differential of</u> $\mathbb{G}$;

ii) <u>If</u> α <u>is the multi-index</u> $\langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$ (1 <u>in the</u> j^{th} <u>place) then the map</u>

$$s_0 s_\alpha + s_\alpha s_0$$

<u>is multiplication by</u> x_j :

iii) <u>If</u> γ <u>is a multi-index with</u> $|\gamma| > 1$, <u>then</u>

$$\sum_{\alpha + \beta = \gamma} s_\alpha s_\beta = 0 .$$

<u>Proof of theorem 1</u>. Beginning with the definition $s_0 = \partial$, we will construct the s_α by induction on $|\alpha|$.

Condition ii) is the assertion that s_α , for $\alpha = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$ (with 1 in the j^{th}-place) is a homotopy for multiplication by x_j. Since $x_j M = 0$, multiplication by x_j is indeed homotopic to 0 on $\mathbb{G}$, so the existence of s_α for $|\alpha| = 1$ satisfying condition ii) is assured.

Now suppose s_α have been constructed for α with $|\alpha| < |\gamma_0|$, for some γ_0. Set

$$\mathcal{E} = - \sum_{\substack{\alpha+\beta=\gamma_0 \\ |\alpha|<\gamma_0 \\ |\beta|<\gamma_0}} s_\alpha s_\beta \ ;$$

we search for a map s_{γ_0} with $s_{\gamma_0} s_0 + s_0 s_{\gamma_0} = \mathcal{E}$, where s_0 is the differential of $\mathbb{G}$. Since $s_0^2 = 0$, any map of the form $ss_0 + s_0 s$ must commute with s_0. A straightford but tedious computation, using i), ii) and iii) with $|\gamma| < |\gamma_0|$, shows that, indeed, $\mathcal{E} s_0 = s_0 \mathcal{E}$. The next lemme thus finishes the proof.

<u>Lemma</u> : <u>Let</u> $\mathbb{G}$ <u>be a free resolution of an A-module</u> M, <u>and suppose that the annihilator of</u> M <u>contains a non zero divisor. Let</u> $\mathcal{E} : \mathbb{G} \longrightarrow \mathbb{G}$ <u>be an endomorphism of degree</u> $k > o$ <u>of</u> $\mathbb{G}$ <u>as a graded module, and let</u> s_0 <u>be the differential of</u> $\mathbb{G}$. <u>Then there exists an endomorphism</u> s <u>of</u> $\mathbb{G}$ <u>as a graded module such that</u> :

$$s_0 s + ss_0 = \mathcal{E}$$

<u>if and only if</u> :

$$s_0 \mathcal{E} = \mathcal{E} s_0$$

<u>Proof of the lemma</u> : The necessity of the condition is obvious ; we prove the sufficiency. Since $\mathcal{E} : \mathbb{G} \longrightarrow \mathbb{G}$ commutes with s_0, it is an endomorphism of $\mathbb{G}$ as a complex of degree $k > o$. But since M is annihilated by a non zero divisor, the induced map :

$$M = \mathrm{Coker}\ (G_1 \longrightarrow G_0) \longrightarrow \mathrm{Coker}(G_{k+1} \longrightarrow G_k) \subset G_{k-1}$$

must be 0. Thus $\mathcal{E}$ is homotopic to 0 ; that is, there exists a map s such that $\mathcal{E} = s_0 s + ss_0$.

Returning to the case in which $x_1, \ldots, x_n$ is an A-sequence, we will use the maps constructed in Theorem 1 (which should be thought of as giving an "approximate" differential graded Λ-module structure) to construct a B-free resolution of M.

<u>Theorem 2</u> : <u>Let</u> A <u>be a ring, and let</u> $x_1, \ldots, x_n$ <u>be an A-sequence. Set</u> $B = \dfrac{A}{(x_1, \ldots, x_n)}$, <u>and let</u> $\mathbb{G}$ <u>be an A-free resolution of a B-module</u> M. <u>Let</u> $\{s_\alpha\}$ <u>be a family of endomorphisms of</u> $\mathbb{G}$ <u>as a graded module satisfying the conditions of Theorem 1. Finally, let</u> $t_1, \ldots, t_n$ <u>be variables of degree</u> -2, <u>and set</u>

$$\mathbb{D} = D(B^n) = \mathrm{Hom}_{\text{graded B-modules}}(B[t_1,\dots,t_n], B),$$

<u>with the natural structure of a</u> $B[t_1,\dots,t_n]$ <u>-module. The graded B-module</u> $\mathbb{D} \otimes \mathbb{G}$, <u>equipped with the differential</u>

$$\partial = \sum_{\alpha} t^{\alpha} \otimes s_{\alpha},$$

<u>is a B-free resolution of</u> M

<u>Remarks</u> : 1) Writing $\tau_1,\dots,\tau_n$ for the dual basis to $t_1,\dots,t_n$, so that $\tau^{(\alpha)} = \tau_1^{(\alpha_1)} \dots \tau_n^{(\alpha_n)}$ is a dual basis of $\mathbb{D}$ to the base of monomials we see that $t^{\alpha}(\tau^{(\beta)}) = 0$ for all α with $|\alpha| > |\beta|$, so that $\partial_{\mathbb{D} \otimes \mathbb{F}}$ is well defined even though the sum involved is formally infinite.

2) It is interesting to compare the construction given in theorem 2 with these are given by Gulliksen in [G] , which may be described as follows : let $\Lambda^+ = \sum_{i>0} \Lambda^i A^n + \partial A^n$ be the "augmentation ideal" of Λ (so that $\Lambda / \Lambda^+ \cong B$). Let $\mathbb{G}$ be a Λ-free resolution of the B-module M, regarded as a Λ-module by the map $\Lambda \longrightarrow B$. Then

$$\overline{\mathbb{G}} = \Lambda / \Lambda^+ \otimes_{\Lambda} \mathbb{G}$$

is a B-free resolution of M.)

Rather than prove Theorem 2, we refer the reader to [E] , and close with a problem :

<u>Problem</u> : Let A be a regular local ring. Which B-modules have minimal resolutions of the form $\mathbb{D} \otimes \overline{\mathbb{G}}$ given in Theorem 2 (on the form $\Lambda / \Lambda^+ \otimes_{\Lambda} G$ given above) ?

REFERENCES

[Av] AVROMOV : огомологии тензорного произведения Izv. Akad. Nauk SSSR - Ser. Mar 39 (1975) (3-14).

[B-E] BUCHSBAUM, D.A. and EISENBUD, D : Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3, to appear

[E] EISENBUD, D. : Homological Algebra on a Complete Intersection, to appear.

[G.L] GULLIKSEN, T., and LEVIN, G. : Homology of local rings, Queen's Papers in Pure and Applied Mathematics n°20, 1969, Queen's University, Kingston, Ontario.

[Lev.] LEVIN, G. : Local rings and Golod homomorphisms J. of Alg. Vol 37, n°2 1975, pp.266-289.

[G] TOR.H. GULLIKSEN. : A change of ring theorem with applications to Poincaré séries and intersection multiplicity. Preprint séries n°15. Institute of Mathematics University of Oslo 1973.