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An algebraic formula for the degree of a C^∞ map germ

By DAVID EISENBUD and HAROLD I. LEVINE*

With an appendix

Sur une inégalité à la Minkowski pour les multiplicités

By BERNARD TEISSIER

If $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ is a continuous map and 0 is isolated in $f^{-1}(0)$, then the degree, $\deg f_0$, of f at 0 is defined as follows: Choose a ball, B_ε , about 0 in \mathbf{R}^n so small that $f^{-1}(0) \cap B_\varepsilon = \{0\}$, and let S_ε be its boundary $(n - 1)$ -sphere. Choose an orientation of each copy of \mathbf{R}^n . The degree at 0 of f is the degree of the mapping $(f/|f|): S_\varepsilon \rightarrow S$, the unit sphere, where the spheres are oriented as $(n - 1)$ -spheres in \mathbf{R}^n . The degree depends only on the germ, f_0 , of f at 0 . If f is differentiable, this degree can be computed as the sum of the signs of the Jacobian of f at all the f -preimages near 0 of a regular value of f near 0 ([17] Lemma 3, p. 36, Lemma 4, p. 37).

If f is a smooth (that is C^∞) map, then another invariant of the germ, f_0 , of f at 0 is its local ring

$$Q(f_0) = C_0^\infty(\mathbf{R}^n)/(f_0),$$

where $C_0^\infty(\mathbf{R}^n)$ is the ring of germs at 0 of smooth real-valued functions on \mathbf{R}^n , and (f_0) is the ideal generated by the components of f_0 . If f_0 is *finite*, in the sense that $Q(f_0)$ is a finite dimensional real vector space, then 0 is isolated in $f^{-1}(0)$, and so the degree of f_0 is defined. Using the notion of contact equivalence it is not hard to show that $|\deg f_0|$, the absolute value of the degree of f_0 , is determined by the ring structure of $Q(f_0)$. The degree itself is not determined by $Q(f_0)$, since its sign depends on the orientations chosen for the two copies of \mathbf{R}^n .

In this paper we will show how to compute $|\deg f_0|$ from $Q(f_0)$, where f_0 is a finite smooth map germ, and we will show how to compute $\deg f_0$

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itself from $Q(f_0)$ and the residue class $J_0 \in Q(f_0)$ of the Jacobian determinant of f_0 : It turns out that $|\deg f_0|$ is the vectorspace dimension of $Q(f_0)$ minus twice the dimension of a maximal square-0 ideal of $Q(f_0)$ (Theorem 1.1) and that $\deg f_0$ is the signature of a certain quadratic form defined on $Q(f_0)$ (Theorem 1.2).

One consequence of our results, together with a result of Teissier proved by him in the appendix to this paper, is that if f is singular then $|\deg f_0| < (1/2) \dim_{\mathbf{R}} Q(f_0)$, and in certain cases it must be still less (Theorem 2.1). This has a curious consequence for the geometry of a real analytic map (Corollary 2.3).

Precise statements of our main results, and an example, are given in Sections 1 and 2.

Section 3 is devoted to an algebraic study of finite dimensional local \mathbf{R} -algebras which, like $Q(f_0)$, support a nonsingular form which is defined by means of a linear functional. These objects are classical (they are variously called symmetric or 0-dimensional Gorenstein \mathbf{R} -algebras), and we briefly review the elementary facts before turning to questions about the signatures of the forms defined on such rings. Later sections make use of this material.

Section 4 contains the proofs of the theorems. Our main tool is a duality result for finite analytic maps (Theorem 4.3); very roughly speaking, it provides us with a canonical nonsingular quadratic form on the ring $Q(f_0)$ (Proposition 4.4).

The appendix, by Bernard Teissier, contains some inequalities on the multiplicities of certain ideals in a local Cohen-Macaulay ring, which are used in Section 3.

We are grateful to Pierre Deligne for his penetrating comments on many aspects of this paper. We also owe a debt to Norbert A'Campo, who first noticed that Theorem 2.1, which we had conjectured in a preliminary version of this paper, could be deduced from an inequality such as Teissier's. (A'Campo's argument, which depends on expressing the degree as a linking number, is not the one we give.)

The question of algebraically computing the topological degree of a finite map germ seems to have been raised by Arnold; a proof of the existence of an algebraic method of computation is given by Zakalyukin in [22]. Unfortunately, that proof does not tell how to carry out the computation.

1. How to compute the degree

THEOREM 1.1. *Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite map germ with local*

ring $Q(f)$. If I is an ideal of $Q(f)$ which is maximal with respect to the property $I^2 = 0$, then

$$|\deg f| = \dim_{\mathbf{R}} Q(f) - 2 \dim_{\mathbf{R}} I.$$

This result will be deduced, using the results of the section on real quadratic forms, from the following theorem which computes the degree itself.

Recall that the signature of a symmetric bilinear form is the sum of the signs of the diagonal entries of any diagonal matrix representing the form.

Our main theorem is

THEOREM 1.2. *Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite map germ, and let $J_0 \in Q(f)$ be the residue class of the Jacobian, J , of f . If $\varphi: Q(f) \rightarrow \mathbf{R}$ is a linear functional such that $\varphi(J_0) > 0$, and if $\langle, \rangle = \langle, \rangle_\varphi$ is the symmetric bilinear form on the ring $Q(f)$ defined by*

$$\langle p, q \rangle = \varphi(pq) \quad \text{for } p, q \in Q(f),$$

Then

$$\deg f = \text{signature } \langle, \rangle.$$

It will turn out that the form \langle, \rangle_φ is nonsingular (Corollary 4.5).

To apply this theorem one must know that there are linear functionals, φ , with $\varphi(J_0) > 0$. That is, one must know $J_0 \neq 0$. This is a consequence of Proposition 4.4 ii. The statement of Theorem 1.2 implies that the signature \langle, \rangle_φ is independent of the choice of φ , as long as $\varphi(J_0) > 0$. We prove this fact separately (Corollary 4.6), and prove Theorem 1.2 using a canonical functional on $Q(f)$, which is defined in Proposition 4.4, in place of φ .

Example: Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}^2: (x, y) \rightarrow (x^2 - y^2, 2xy)$. (If we identify \mathbf{R}^2 with \mathbf{C} , this is just the map $z \rightarrow z^2$.) A basis for $Q(f)$ is given by the residue classes of $1, x, y, 4(x^2 + y^2) = J$. If φ is the functional sending J_0 to $1 \in \mathbf{R}$ and sending the other basis elements to 0 , then the matrix of \langle, \rangle_φ with respect to this basis is:

$$\begin{matrix} & \begin{matrix} 1 & x & y & J \end{matrix} \\ \begin{matrix} 1 \\ x \\ y \\ J \end{matrix} & \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 1/8 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right) \end{matrix}$$

This matrix has signature 2, which is the degree of f at 0 .

2. A geometric consequence

Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite map germ. In this section we will

compare the degree of f with $\dim_{\mathbf{R}} Q(f)$, which is called the multiplicity of f . Write f again for a mapping representing f , and let y be a regular value of f near 0. Writing $\#X$ for the number of elements in a set X , we have by [8], Prop. 2.4, p. 168,

$$|\deg f| \leq \#f^{-1}(y) \leq \dim Q(f).$$

If f is singular at 0, then $Q(f)$ is an Artinian local ring which is not a field, so $Q(f)$ contains nonzero ideals with square 0 (for instance, the largest power of the maximal ideal which is nonzero must have square 0). Thus from Theorem 1.1 we see that the degree of f must be strictly less than $\dim Q(f)$. Actually, much stronger inequalities hold:

THEOREM 2.1. *Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite map germ. Then*

- i. $|\deg f| \leq (\dim Q(f))^{1-1/n}$
- ii. *If f is singular, then $|\deg f| < (1/2) \dim Q(f)$.*

Theorem 2.1 will be proved in Section 4, using Corollary 3.10, which relies on Teissier's appendix.

To get at the geometric content of the above results, we first recall the classical interpretation of the multiplicity of a complex analytic map germ. If \mathbf{D} is a domain of \mathbf{C}^n and $g: (\mathbf{D}, x) \rightarrow (\mathbf{C}^n, y)$ is a complex analytic map, we will write $Q_{\mathbf{C}}(g_x)$ for the ring of complex-analytic germs at x modulo the ideal generated by the components of g_x .

If $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ is an analytic map germ, let $f_{\mathbf{C}}$ be the complex analytic mapping defined by the same power series which define f .

PROPOSITION 2.2. *Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite analytic map germ, and suppose that the power series representing f converge on a disc $\mathbf{D} \subset \mathbf{C}^n$ about 0, that is, $f_{\mathbf{C}}$ defines a complex analytic map of \mathbf{D} into \mathbf{C}^n . Suppose that \mathbf{D} has been chosen small enough so that $f_{\mathbf{C}}^{-1}(0) \cap \mathbf{D} = \{0\}$. Then there is a neighborhood U of 0 in \mathbf{R}^n such that for all $y \in U$*

$$\dim_{\mathbf{R}} Q(f) = \sum_{x \in f_{\mathbf{C}}^{-1}(y) \cap \mathbf{D}} \dim Q_{\mathbf{C}}(f_{\mathbf{C},x}).$$

In particular, for almost all $y \in U$, $\dim_{\mathbf{R}} Q(f) = \#(f_{\mathbf{C}}^{-1}(y) \cap \mathbf{D})$.

Proof. We clearly have $Q_{\mathbf{C}}(f_{\mathbf{C},0}) \cong \mathbf{C} \otimes Q(f)$, so $\dim_{\mathbf{R}} Q(f) = \dim_{\mathbf{C}} Q_{\mathbf{C}}(f_{\mathbf{C},0})$. The first statement follows from [9], Corollary 2 to Theorem 21, page 137. The second statement is a consequence of the first since the regular values of $f_{\mathbf{C}}$ contain open dense sets in \mathbf{C}^n and in \mathbf{R}^n .

Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite analytic map germ, and let $\mathbf{D} \subset \mathbf{C}^n$ and $U \subset \mathbf{R}^n$ be as in Proposition 2.2. For $y \in U$, set

$$\text{re } f^{-1}(y) = f^{-1}(y) \cap \mathbf{D}$$

($\text{re } f^{-1}(y)$ is the set of real preimages of y close to 0) and

$$\text{im } f^{-1}(y) = (f_c^{-1}(y) \cap \mathbf{D}) - \text{re } f^{-1}(y) .$$

We have just seen that if y is a regular value for f_c , then

$$\dim_{\mathbf{R}} Q(f) = \# \text{re } f^{-1}(y) + \# \text{im } f^{-1}(y) .$$

On the other hand, the degree of f is obtained by counting the points in $\text{re } f^{-1}(y)$ as plus or minus 1, depending on whether f preserves or reverses the orientation at those points. Thus we may write $|\text{deg } f| = \# \text{re } f^{-1}(y) - 2m$, where

$$m = \min \left(\# \{x \in \text{re } f^{-1}(y) \mid f \text{ preserves the orientation at } x\} , \right. \\ \left. \# \{x \in \text{re } f^{-1}(y) \mid f \text{ reverses the orientation at } x\} \right) .$$

Theorem 2.1 may be considered as giving a lower bound for m in terms of $\# \text{re } f^{-1}(y)$ and $\# \text{im } f^{-1}(y)$:

COROLLARY 2.3. *If, with notations as above, $y \in U$ is a regular value of f_c , then*

$$\# \text{re } f^{-1}(y) - |\text{deg } f| \geq (1/2)(\# \text{re } f^{-1}(y) - \# \text{im } f^{-1}(y)) .$$

Proof of Corollary 2.3. The absolute value of the degree of f is

$$|\text{deg } f| = \# \text{re } f^{-1}(y) - 2m ,$$

but

$$m = (1/2)(\# \text{re } f^{-1}(y) - |\text{deg } f|) \\ \geq (1/2)(\# \text{re } f^{-1}(y) - (1/2) \dim Q(f)) \quad (\text{by Theorem 2.1, ii}) \\ = (1/2)(\# \text{re } f^{-1}(y) - (1/2)(\# \text{re } f^{-1}(y) + \# \text{im } f^{-1}(y))) \quad (\text{by Proposition 2.2}) \\ = (1/4)(\# \text{re } f^{-1}(y) - \# \text{im } f^{-1}(y)) ,$$

as required. //

A similar argument yields a result corresponding to Theorem 2.1, i.

It is easy to see that Theorem 2.1 gives the best possible bound in the case $n = 1$. The next proposition shows in particular that this remains true for $n = 2$.

PROPOSITION 2.4. *If $g: (C^n, 0) \rightarrow (C^n, 0)$ is a finite complex analytic map germ, and if $f: (\mathbf{R}^{2n}, 0) \rightarrow (\mathbf{R}^{2n}, 0)$ is the germ obtained from g by identifying \mathbf{R}^{2n} with C^n , then*

$$(\text{deg } f)^2 = \dim_{\mathbf{R}} Q(f) .$$

Proof. As in the proof of Proposition 2.2, we write

$$Q_c(g) = C\{z\}/(g) ,$$

where (g) is the ideal generated by the components g_1, \dots, g_n of g .

Since \mathbf{C}^n has a canonical orientation (as a real $2n$ manifold), all the pre-images of a regular value of g count positively in the calculation of $\deg f$ and as in Proposition 2.2, the number of such pre-images close enough to 0 is $\dim Q_{\mathbf{C}}(g)$. Thus

$$\deg f = \dim_{\mathbf{C}} Q_{\mathbf{C}}(g) .$$

Define \bar{g} by $\bar{g}(\bar{z}) = \overline{g(z)}$. The components of f can be written $(g(z) + \bar{g}(\bar{z}))/2$ and $(g(z) - \bar{g}(\bar{z}))/2i$. From this it follows that

$$\mathbf{C} \otimes Q(f) \cong \mathbf{C}\{z\}/(g) \otimes_{\mathbf{C}} \mathbf{C}\{\bar{z}\}/(\bar{g}) .$$

Since $\dim_{\mathbf{C}} \mathbf{C}\{z\}/(g) = \dim_{\mathbf{C}} \mathbf{C}\{\bar{z}\}/(\bar{g})$, we have

$$\dim_{\mathbf{R}} Q(f) = \dim_{\mathbf{C}} (\mathbf{C} \otimes Q(f)) = (\dim_{\mathbf{C}} Q_{\mathbf{C}}(g))^2 = (\deg f)^2 . \quad //$$

We close with two higher dimensional examples for which the degree is relatively large:

Example. Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be the map germ whose coordinate functions are

$$\begin{aligned} f_1 &= \prod_1^n x_i , \\ f_j &= x_1^2 - x_j^2 \end{aligned} \quad \text{for } j = 2, \dots, n .$$

Computation shows that

$$\begin{aligned} |\deg f| &= 2^{n-1} \\ \dim_{\mathbf{R}} Q(f) &= n2^{n-1} . \end{aligned}$$

Thus in this case, the bound of Theorem 2.1, ii could be improved to $|\deg f| \leq (1/n) \dim Q(f)$. That is not so in general, even for map germs whose components are all in the square of the maximal ideal of $C_0^\infty(\mathbf{R}^n)$, as shown by the next example:

Example. Let $f: \mathbf{R}^4 \rightarrow \mathbf{R}^4$ be the map germ whose coordinate functions are

$$\begin{aligned} f_1 &= x_1^2 + \sum_{0 \leq i < j \leq 4} x_i x_j , \\ f_j &= x_1^2 - x_j^2 , \end{aligned} \quad j = 2, 3, 4 .$$

Computation shows that

$$|\deg f| = 6 ,$$

but

$$\dim_{\mathbf{R}} Q(f) = 16 .$$

3. The Structure of an algebra with a non-singular form

Throughout this section, F is a field of characteristic $\neq 2$, Q is commutative local F -algebra that is a finite dimensional F -vector space, and φ

is a linear functional such that the symmetric bilinear form $\langle, \rangle = \langle, \rangle_\phi$ defined by

$$\langle p, q \rangle = \phi(pq) \text{ for } p, q \in Q$$

is non-singular.

If f is a finite map germ, and if $\phi: Q(f) \rightarrow \mathbf{R}$ satisfies $\phi(J_0) > 0$ as in Theorem 1.2, then $F = \mathbf{R}$, $Q = Q(f)$, and ϕ satisfy the above conditions, as we shall see in Proposition 4.3 (v).

Classically finite dimensional (not necessarily commutative) algebras over an arbitrary field which support non-singular forms defined by functionals as above were studied under the name *symmetric algebras*. This notion was originally investigated by Nakayama because the group algebra of a finite group is symmetric, no matter what the characteristic of the ground field is. (See [7], Chapter IX.) In the commutative case they are often called 0-dimensional Gorenstein rings. See [3].

We begin with some classical results, always restricting ourselves to the commutative case.

If S is any F -algebra, then $\text{Hom}_F(S, F) = S^*$ is an S -module by:

$$(q\psi)(p) = \psi(qp), \text{ for } \psi \in S^*, p, q \in S.$$

PROPOSITION 3.1. *Let S be a finite dimensional F -algebra, and suppose $\psi \in S^*$. Then \langle, \rangle_ψ is non-singular if and only if ψ generates S^* as an S -module. (Thus there exists a functional $\psi \in S^*$ such that \langle, \rangle_ψ is non-singular if and only if $S^* = S$ as S -modules.)*

Proof. The form \langle, \rangle_ψ is non-singular if and only if $0 = \langle s, - \rangle_\psi$ implies $s = 0$ if and only if the map $S \rightarrow S^*: s \rightarrow s\psi$ is an injection. Since S is finite dimensional, a map $S \rightarrow S^*$ is an injection if and only if it is an epimorphism. //

PROPOSITION 3.2. *Let I be an ideal of Q . Then*

- (i) $\text{Ann}_Q(I) = I^\perp$; in particular, I^\perp is an ideal.
- (ii) *The lattice of ideals of Q is self dual, the duality sending an ideal to its annihilator.*

Proof. Since \langle, \rangle is non-singular, $p \in \text{Ann}_Q I$ if $\langle pI, Q \rangle = 0$. But $\langle pI, Q \rangle = \phi(pIQ) = \langle p, I \rangle$. Thus $\text{Ann}_Q I = I^\perp$. The rest is immediate. //

We will write \mathfrak{M} for the maximal ideal of Q .

COROLLARY 3.3. *Q has a unique minimal ideal.*

Proof. The unique minimal ideal is just the annihilator of \mathfrak{M} , the unique maximal ideal. //

In any local algebra, the annihilator of the maximal ideal is called the

socle. We see from the proof of the corollary that the socle of Q is its unique minimal ideal.

PROPOSITION 3.4. *If $\psi = Q \rightarrow F$ is a functional, then \langle, \rangle_ψ is singular if and only if the restriction of ψ to the socle of Q is identically 0.*

Proof. Since the socle is an ideal, $\langle \text{socle}, Q \rangle_\psi = \psi(\text{socle})$; so if ψ vanishes on the socle, the form is singular. Conversely, if \langle, \rangle_ψ is singular, then $0 = \langle q, Q \rangle_\psi = \psi(qQ)$ for some $q \in Q$. But qQ contains the socle, since the socle is the unique minimal ideal.

We now fix a nonzero element J_0 of the socle of Q such that $\varphi(J_0) \neq 0$.

In the case of primary interest, with $F = \mathbf{R}$, the form \langle, \rangle_φ does not depend too heavily on φ . This is explained by the next result. Note that if $F = \mathbf{R}$, then Q/\mathfrak{M} is \mathbf{R} or \mathbf{C} .

PROPOSITION 3.5. *Let $\psi \in Q^*$ be such that \langle, \rangle_ψ is nonsingular. Suppose that either*

- (i) $Q/\mathfrak{M} = F$ and $\varphi(J_0)\psi(J_0)$ is a square in F , or
- (ii) Every element of Q/\mathfrak{M} has a square root.

Then the forms \langle, \rangle_φ and \langle, \rangle_ψ are equivalent.

The following proof was pointed out to us by P. Deligne:

Proof. In either case, Proposition 3.1 shows that φ and ψ are both generators of the Q -module Q^* , so there is a unit p of Q such that $p\varphi = \psi$. We claim that the image \bar{p} of p in Q/\mathfrak{M} is a square. In case ii, this is automatic, while in case i we may regard \bar{p} as an element of F , so we have

$$\begin{aligned} \psi(J_0) &= (p\varphi)(J_0) \\ &= \varphi(pJ_0) \\ &= \varphi(\bar{p}J_0) && \text{(since } \mathfrak{M}J_0 = 0) \\ &= \bar{p} \cdot \varphi(J_0) . \end{aligned}$$

Since $\varphi(J_0)\psi(J_0)$ is a square, so is \bar{p} .

Say $\bar{p} = r^2$, with $r \in F$, so that $r^{-2} \cdot p = 1 + n$, with $n \in \mathfrak{M}$. Since n is nilpotent, and 2 is a unit in Q , the binomial formula yields a square root p' of $1 + n$ in Q , and rp' is a square root of p .

We claim that multiplication by rp' is an equivalence between \langle, \rangle_φ and \langle, \rangle_ψ , for

$$\begin{aligned} \langle rp'a, rp'b \rangle_\varphi &= \varphi(r^2 p'^2 ab) \\ &= \varphi(pab) \\ &= p\varphi(ab) \\ &= \psi(ab) \\ &= \langle a, b \rangle_\psi . \quad // \end{aligned}$$

We now turn to the structure of the quadratic form \langle, \rangle_ϕ . Recall that a subspace $I \subset Q$ is called isotropic if $\langle I, I \rangle_\phi = 0$. Since \langle, \rangle is non-singular and the characteristic of F is not 2, Q can be decomposed as

$$Q \cong I \oplus I^* \oplus D \text{ as vectorspaces,}$$

where I is any maximal isotropic subspace, I^* is dual to I under the pairing \langle, \rangle_ϕ , and $D = (I \oplus I^*)^\perp$. The inner product space D is anisotropic (contains no nonzero isotropic subspaces).

PROPOSITION 3.6. *Suppose $Q/\mathfrak{M} = F$, and let I be an ideal of Q which is maximal with respect to the property $I^2 = 0$. Then I is a maximal isotropic subspace of Q .*

Proof. Since $\langle I, I \rangle_\phi = \phi(I^2) = 0$, I is isotropic. Suppose for some $q \notin I$, that $I \oplus Fq$ is isotropic. If $\mathfrak{M}q \subset I$, then $I \oplus Fq$ would be an ideal, so since $\langle I \oplus Fq, I \oplus Fq \rangle_\phi = 0$, Proposition 3.2 shows $(I \oplus Fq)^2 = 0$, contradicting the maximality of I . Otherwise, let $p \in \mathfrak{M}$ be such that $pq \notin I$ but $pq\mathfrak{M} \subseteq I$; this is always possible by the finite dimensionality of Q . Clearly, for this choice of p , $I' = I \oplus Fpq$ is an ideal. We obtain a contradiction as above by showing that I' is isotropic.

We have

$$\langle I', I' \rangle = \langle I, I \rangle + \langle I, pq \rangle + \langle pq, pq \rangle .$$

Since $I \oplus Fq$ is isotropic, $\langle I, I \rangle = 0$, and $\langle I, pq \rangle = \langle Ip, q \rangle \subseteq \langle I, q \rangle = 0$. Finally,

$$\langle pq, pq \rangle = \langle p^2q, q \rangle \subset \langle \mathfrak{M}pq, q \rangle \subset \langle I, q \rangle = 0 . //$$

COROLLARY 3.7. *Suppose $F = \mathbf{R}$:*

- (i) *If $Q/\mathfrak{M} = \mathbf{C}$, then signature $\langle, \rangle_\phi = 0$.*
- (ii) *If $Q/\mathfrak{M} = \mathbf{R}$, and I is a maximal square-zero ideal of Q , then*

$$|\text{signature } \langle, \rangle_\phi| = \dim_{\mathbf{R}} Q - 2 \dim_{\mathbf{R}} I .$$

Proof. Decompose Q as $I \oplus I^* \oplus D$, with I a maximal isotropic subspace. D will be (positive or negative) definite, and $|\text{signature } \langle, \rangle_\phi| = \dim_{\mathbf{R}} D$. In case (ii) I may be taken to be any maximal square-0 ideal, by Proposition 3.6, and the conclusion follows. In case (i) Proposition 3.5 (ii) shows that \langle, \rangle_ϕ is isometric with $\langle, \rangle_{-\phi}$. However, these two forms restricted to D are definite of opposite sign. Since the restriction of the forms to D is uniquely determined, $D = 0$, as required. //

Now suppose $Q/\mathfrak{M} = F$, and let I be a maximal square zero ideal of Q . By Proposition 3.6, I is a maximal isotropic subspace so

$$Q = I \oplus I^* \oplus D, \text{ as } \mathbf{R}\text{-vectorspaces,}$$

where the form is definite on D .

PROPOSITION 3.8. *With the above decomposition, the subspace $I \oplus D$ is an ideal in Q , and $\mathfrak{M}D \subset I$.*

Proof. Since $I \oplus D = I^\perp$, the first conclusion follows from Proposition 3.2(i). Let k be the least integer such that $\mathfrak{M}^k D \subseteq I$, and assume that $k > 1$. Then

$$\langle \mathfrak{M}^{k-1} D, \mathfrak{M}^{k-1} D \rangle = \langle \mathfrak{M}^k D, \mathfrak{M}^{k-2} D \rangle \subseteq \langle I, I \oplus D \rangle = 0$$

since $I \oplus D$ is an ideal. Thus $\mathfrak{M}^{k-1} D$ is an isotropic subspace. But if $p \in \mathfrak{M}^{k-1} D \subseteq I \oplus D$, then writing $p = q + d$ with $q \in I$ and $d \in D$, we have $0 = \langle p, p \rangle = \langle d, d \rangle$. So $d = 0$ and $p \in I$, a contradiction. //

The most important rings for us are those of the form $Q(f)$, where $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a finite map germ. They are not only Gorenstein, but can even be represented in the form:

$$Q = F[[X_1, \dots, X_n]]/\mathfrak{A}$$

where \mathfrak{A} is an ideal generated by n elements. A finite dimensional F -algebra which can be written in this form (for some n) is called a *complete intersection*; it will be automatically Gorenstein ([13], note in Ex. 5, Page 163, see also Section 4C, below) and have $Q/\mathfrak{M} = F$. Our main result concerns complete intersections, to which we can apply Teissier's inequality I. 2 (see appendix).

THEOREM 3.9. *Suppose $Q = F[[X_1, \dots, X_n]]/(f_1, \dots, f_n)$ is a complete intersection, and that*

$$Q = I \oplus I^* \oplus D$$

is a decomposition with I isotropic and D anisotropic for \langle, \rangle_φ . Then:

(i) $\dim_F(D) \leq (\dim_F Q)^{1-1/n}$.

If Q is not a field, then:

(ii) $\dim_F(D) \leq (1/2) \dim_F Q$.

COROLLARY 3.10. *Suppose that $F = \mathbf{R}$ and that Q is a complete intersection as above. Then:*

(i) $|\text{signature } \langle, \rangle_\varphi| \leq (\dim_{\mathbf{R}} Q)^{1-1/n}$.

If Q is not a field, then:

(ii) $|\text{signature } \langle, \rangle_\varphi| \leq (1/2) \dim_{\mathbf{R}} Q$.

Proof of the Corollary. Immediate.

Proof of Theorem 3.9 (i). By Proposition 3.8, together with Nakayama's lemma, the dimension of D is bounded by the number of elements required to generate the ideal $I \oplus D$ of Q . We will show that the number of generators required for any ideal in Q is bounded by $(\dim_F Q)^{1-1/n}$.

Let $\mathfrak{A} = (f_1, \dots, f_n)$. Extending F if necessary, we may assume that F is infinite, and following the notation of the appendix, we let $\mathfrak{A}^{[n-1]}$ be the ideal of $F[[X_1, \dots, X_n]]$ generated by $n - 1$ sufficiently general linear combinations of f_1, \dots, f_n . Set $Q' = F[[X, \dots, X_n]]/\mathfrak{A}^{[n-1]}$. Since the $n - 1$ generators of $\mathfrak{A}^{[n-1]}$ form a regular sequence, Q' is a Cohen-Macaulay local ring of dimension 1. By Teissier's inequality I. 2, applied to $n_1 = \mathfrak{A}$, $n_2 = (X_1, \dots, X_n)$, $i = n - 1$ and $d - i = 1$,

$$e(\mathfrak{A}^{[n-1]} + (X_1, \dots, X_n)^{[1]}) \leq e(\mathfrak{A})^{(n-1)/n},$$

where e is the multiplicity (see, for example, [18] for an elementary exposition of the multiplicity). However, $e(\mathfrak{A}) = \dim_F Q$, since \mathfrak{A} is generated by a regular sequence [20]. Also $e(\mathfrak{A}^{[n-1]} + (X_1, \dots, X_n)^{[1]})$ is the multiplicity of the local ring Q' . Thus we have

$$\text{mult } Q' \leq (\dim_F Q)^{1-1/n}.$$

To complete the proof, we simply note that the number of generators of an ideal in Q is bounded by the number of generators of the preimage of that ideal under the projection $Q' \rightarrow Q$, and apply the following result:

PROPOSITION 3.10. ([23] Theorem 12.8). *If Q' is a 1-dimensional Cohen-Macaulay ring of multiplicity e , then any ideal of Q' can be generated by e elements.*

Proof sketch. We restrict ourselves to the case required for Theorem 3.9. Let X be a sufficiently general linear combination of X_1, \dots, X_n and regard Q' as an $F[[X]]$ -module. Since Q' is Cohen-Macaulay, Q' is a free $F[[X]]$ -module of rank equal to $\dim_F Q'/XQ' = e$. Any ideal of Q' , being an $F[[X]]$ -submodule, will be free of rank $\leq e$. Thus it can be generated—even over $F[[X]]$ —by e elements. //

Proof of Theorem 3.9 (ii). Replacing the representation of Q as a quotient of $F[[X_1, \dots, X_n]]/\mathfrak{A}$ by another, possibly with fewer variables, we may assume that all the n generators of \mathfrak{A} are contained in $(X_1, \dots, X_n)^2$. It follows from a multiplicity argument [18], that $\dim_F Q \geq 2^n$. Using part (i), we thus have

$$\begin{aligned} \dim_F D &\leq (\dim_F Q)^{1-1/n} \\ &= (\dim_F Q)^{-1/n} \dim_F Q \\ &\leq (1/2) \dim_F Q. \end{aligned} //$$

For some examples of complete intersections where D is relatively large, see the end of Section 2 (but note that these examples are not enough to show that Theorem 3.9 is best possible for all n). For Gorenstein Q which are not complete intersections, one has examples like the following:

Example (R. Stanley). Let $Q = \mathbf{R}[[X_1, \dots, X_n]]/\mathfrak{A}$, where \mathfrak{A} is generated by the $\binom{n}{2} + (n-1)$ elements

$$X_i X_j, X_1^2 - X_j^2, 1 \leq i < j \leq n.$$

Q is a local Gorenstein algebra of dimension $n+2$. If φ is a generator of Q^* , and D the corresponding anisotropic space, then $\dim_{\mathbf{R}} D = n$. To obtain an inequality on $\dim_F D$ in general, one can use a result [14] which says that if \mathfrak{A} is an ideal of $F[[X_1, \dots, X_n]]$ such that $F[[X_1, \dots, X_n]]/\mathfrak{A} = Q$ is finite dimensional over F , then $e(\mathfrak{A}) \leq n! \dim_F Q$. Together with the result of Teissier and Proposition 3.10, this yields, in case Q is a Gorenstein ring, and D is its anisotropic subspace as above, the inequality

$$\dim_F D \leq (n! \dim_F Q)^{1-1/n},$$

a result which is, unfortunately, trivial unless

$$\dim Q > (n!)^{n-1}.$$

We close this section with a more optimistic possibility.

Question. Suppose $Q = F[[X_1, \dots, X_n]]/\mathfrak{A}$ is a 0-dimensional Gorenstein ring, and let D be an anisotropic subspace as in Proposition 3.8. Suppose \mathfrak{A} can be generated by m elements.

Is it true that $\dim_F D \leq (\dim_F Q)^{1-1/m}$?

4. Proofs of the theorems

In this section we give the proof of the main theorem, Theorem 1.2. Along the way (Corollary 4.5) we will show that the algebra $Q = Q(f)$, together with a linear functional φ as in Theorem 1.2, satisfies the conditions of Sections 3. This being the case, Theorem 1.1 follows from Theorem 1.2 and Corollary 3.7 while Theorem 2.1 follows from Theorem 1.2 and Corollary 3.10.

In order to prove Theorem 1.2 we will first replace f with an analytic map. We will then define an analytic family of algebras Q_y over \mathbf{R} , all of the same dimension, parametrized by the points y of a small neighborhood of the origin in the target \mathbf{R}^n of f , such that $Q_0 = Q(f)$.

Although it would be possible to extend φ to a family of linear functionals φ_y on Q_y , and even to prove that $\langle \cdot, \cdot \rangle_{\varphi_y}$ are all nonsingular and have constant signature, we know of no direct way to calculate this signature. We therefore appeal to duality theory to define a canonical functional $T_0: Q(f) \rightarrow \mathbf{R}$ which extends canonically to an analytically varying family of functionals $T_y: Q_y \rightarrow \mathbf{R}$. The good properties of this canonical functional make it easy to show that signature $\langle \cdot, \cdot \rangle_{T_y}$ is constant in y and to compute this signature when y is a regular value of f .

4A. Reduction to the analytic case

For the purpose of proving Theorem 1.2, we can replace f by any map g as long as the ideals (f) and (g) in $C_0^\infty(\mathbf{R}^n)$ are equal, their Jacobians, $J(f)$ and $J(g)$ are congruent modulo these ideals, and the degrees of f and g agree.

PROPOSITION 4.1. *Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be finite. If for sufficiently large k , the map $g: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ agrees with f up to k^{th} order, then*

- (i) $(f) = (g)$,
- (ii) $J(f) \equiv J(g)$ modulo (f) ,
- (iii) $\deg f = \deg g$.

Proof. (i) Let \mathfrak{M} be the maximal ideal in $C_0^\infty(\mathbf{R}^n)$. Since f is finite, $(f) \supseteq \mathfrak{M}^k$ for some k . Let g be any map germ whose k -jet at 0, $j^k g(0)$ equals $j^k f(0)$. Then

$$\mathfrak{M}^k \subseteq (f) \subseteq (g) + \mathfrak{M}^{k+1}.$$

By Nakayama's Lemma $(g) \supseteq \mathfrak{M}^k$ so $(f) \subseteq (g)$. Thus $(f) = (g)$.

(ii) If $j^k f(0) = j^k g(0)$, then $j^{k-1}(J(f))(0) = j^{k-1}(J(g))(0)$. So $J(f) \equiv J(g)$ modulo \mathfrak{M}^k . If $(f) \supseteq \mathfrak{M}^k$, then $J(f) \equiv J(g)$ modulo (f) .

(iii) If $(f) = (g)$, we know that there is an $n \times n$ matrix H of C^∞ -map germs such that $f(x) = H(x) \cdot g(x)$ where $H(0)$ is non-singular ([15], Prop. of Sec. 2.3, or [8], Theorem 3.3, Chap. VII). Write $H(x) = H(0) + H_1(x)$. For a sufficiently small neighborhood V of 0 in \mathbf{R}^n , $\|H(0)^{-1}H_1(x)\| < 1$, so clearly $H(0) + tH_1(x)$ is non-singular for any $t \in [0, 1]$ and $x \in V$. Thus $\deg f = \deg(H(0) \cdot g) = \text{sgn}(\det(H(0))) \cdot \deg g$. Now suppose that $j^k f(0) = j^k g(0)$ and $(f) = (g) \supset \mathfrak{M}^k$. We will show that $H(0) = I$, the identity, which will prove (iii).

We have $(f_i - g_i) \in \mathfrak{M}^{k+1} \subseteq (g)\mathfrak{M}$; we write $f \equiv g((g)\mathfrak{M})$. This, together with the equation $f = H \cdot g$ gives $g \equiv H(0) \cdot g((g)\mathfrak{M})$, or $(H(0) - I)g \equiv 0((g)\mathfrak{M})$. If $H(0) \neq I$, this would imply that there is a proper subset $\{g'\}$ of $\{g_1, \dots, g_n\}$ such that $(g') + (g)\mathfrak{M} = (g)$. By Nakayama's Lemma we would have $(g') = (g)$. Thus $(g') = (g) \supseteq \mathfrak{M}^k$, so the image of (g') in the ring of formal power series would contain a power of the maximal ideal there. But no ideal generated by fewer than n elements can have that property ([16], 12. I, p. 77). //

Proposition 4.1 shows that for the purpose of proving Theorems 1.1, 1.2, and 2.1 we may assume that f is the germ of an analytic (even a polynomial) map. Notice that, in this case, by the finiteness of f we have

$$Q(f) = \mathbf{R}[[X]]/(f) = \mathbf{R}\{X\}/(f)$$

where $\mathbf{R}\{X\}$ is the ring of convergent power series.

4B. The family of algebras

We will assume by the grace of Proposition 4.1 that f is a finite real-analytic germ. We will also write f for the unique holomorphic map, defined on a neighborhood of the origin in \mathbb{C}^n , which restricts to the germ f .

Choosing coordinates X_i on the source and Y_i on the target and writing $\mathbf{R}\{X\}$ and $\mathbf{R}\{Y\}$ for the rings of convergent power series in X_1, \dots, X_n and Y_1, \dots, Y_n , respectively, the germ f corresponds to a map of rings

$$f^*: \mathbf{R}\{Y\} \longrightarrow \mathbf{R}\{X\}$$

sending Y_i to the i^{th} coordinate f_i of f . By the Weierstrass preparation theorem [12], I, the finiteness of f implies that $\mathbf{R}\{X\}$ becomes in this way an $\mathbf{R}\{Y\}$ -algebra which is finitely generated as an $\mathbf{R}\{Y\}$ -module. Such an $\mathbf{R}\{Y\}$ -algebra is said to be *finite*.

To obtain a more global result, we sheafify. Write \mathcal{O} for the sheaf of real analytic functions on the target \mathbf{R}^n .

THEOREM 4.2. *Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a finite analytic map germ. There exist neighborhoods U of 0 in the source and V of 0 in the target and a coherent sheaf of finite \mathcal{O}_V -algebras \mathcal{F} such that*

- (i) \mathcal{F} is a free \mathcal{O}_V -module (with rank equal to $\dim_{\mathbf{R}} \mathcal{Q}(f)$);
- (ii) $\mathcal{F}_0 \cong \mathbf{R}\{X\}$ as an $\mathcal{O}_0 = \mathbf{R}\{Y\}$ -algebra;
- (iii) f is represented by an analytic map on U .

Furthermore, the map f restricted to U is isomorphic to the natural map $(\text{Specan } \mathcal{F}) \rightarrow V$.

In particular:

- (iv) If x_1, \dots, x_k are the preimages in U of a point y in V , then $\mathcal{F}_y = \mathcal{F}_y e_1 \times \dots \times \mathcal{F}_y e_{\text{complex}}$ (product of algebras) where $\mathcal{F}_y e_i$ is the ring of germs of real analytic functions at x_i , regarded as an \mathcal{O}_y algebra by means of f^* , and $\mathcal{F}_y e_{\text{complex}}$ is a product of finitely many local rings, each of which is isomorphic as a ring to $\mathbb{C}\{X_1, \dots, X_n\}$.

Remark. In fact, the local direct factors of $\mathcal{F}_y e_{\text{complex}}$ correspond to pairs of conjugate complex points in $f^{-1}(y)$. The algebra \mathcal{F} can be constructed directly as follows: There is a neighborhood U_c of U in \mathbb{C}^n such that the holomorphic function f is defined on U_c , and such that the sections of \mathcal{F} over an open set $V' \subset V$ are the holomorphic functions on $f^{-1}(V') \cap U_c$ which commute with complex conjugation.

References in place of a proof. For all except statement (i), see [12], II, Corollaries 1 and 2 of Proposition 6 (Corollary 2 says that analytic local rings are henselian; for part (iv) we have used the property of henselian

rings given by Houzel in part (ii) of the theorem in the appendix to [12] II.)

As for part (i), it suffices to prove the freeness of $\mathcal{F}_0 = \mathbf{R}\{X\}$ over $\mathcal{O}_0 = \mathbf{R}\{Y\}$, since if a coherent \mathcal{O}_V module is free at a point, then it is free in a whole neighborhood (see, for instance, [10] Chapter 4, B6 and C3). The freeness of $\mathbf{R}\{X\}$ as an $\mathbf{R}\{Y\}$ -module may be treated exactly as in [9], Corollary 2 to Theorem 21 (or see [1], Page 53, Proposition 3.16 and [20], Proposition 21, Page IV-35.)

To obtain the promised family of algebras, we let \mathfrak{M} be the sheaf of maximal ideals of \mathcal{O} , and for $y \in V$, we define

$$\begin{aligned} Q_y &= \mathcal{F}_y / \mathfrak{M}_y \mathcal{F}_y, \\ Q_0 &= \mathcal{F}_0 / \mathfrak{M}_0 \mathcal{F}_0 \\ &= \mathbf{R}\{X\} / (f^*(y_1), \dots, f^*(y_n)) \mathbf{R}\{X\} \\ &= \mathbf{R}\{X\} / (f_1, \dots, f_n) \\ &= Q(f). \end{aligned}$$

If we let a_1, \dots, a_m be a basis of the free \mathcal{O}_V -module \mathcal{F} , then the images of a_1, \dots, a_m will form a basis of the $\mathbf{R} = \mathcal{O}_y / \mathfrak{M}_y$ -vector space $Q_y = \mathcal{F}_y / \mathfrak{M}_y \mathcal{F}_y$ for every $y \in V$, so $\dim_{\mathbf{R}} Q_y = m$ for every such y .

4C. Duality

We will keep the hypotheses and notations of Theorem 4.2. Let \mathcal{K} be the sheaf of meromorphic functions on \mathbf{R}^n , and let \mathcal{L} be the sheaf of rings of total quotients of \mathcal{F} . For each point $y \in V$, \mathcal{L}_y is the product of the quotient fields of the local direct factors of \mathcal{F}_y , and thus is a finite product of finite field extensions of \mathcal{K}_y . Thus there is a trace map $\text{Tr}: \mathcal{L}_y \rightarrow \mathcal{K}_y$ which sheafifies to a map $\text{Tr}: \mathcal{L} \rightarrow \mathcal{K}_V$. Given any open set $V' \subset V$ and sections $p \in \mathcal{L}(V')$ and $\psi \in \text{Hom}_{\mathfrak{M}_V}(\mathcal{L}, \mathcal{K}_V)(V')$, we may define $p\psi$ by the formula $(p\psi)(b) = \psi(pb)$. As in classical theory of finite separable field extensions, this makes $\text{Hom}_{\mathcal{K}_V}(\mathcal{L}, \mathcal{K}_V)$ into an \mathcal{L} -module isomorphic to \mathcal{L} , generated by the global section Tr (see [21]). The duality result that we need computes $\text{Hom}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{O}_V)$ in a similar way:

THEOREM 4.3. *Let X_i and Y_i be coordinates on U and V , and let $J \in \mathcal{F}(V)$ be the section corresponding to the Jacobian determinant $\partial(f_1, \dots, f_n) / \partial(X_1, \dots, X_n)$. Then, possibly after V is replaced by a smaller neighborhood of 0 , the sheaf*

$$\text{Hom}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{O}_V)$$

is isomorphic as an \mathcal{F} -module, to \mathcal{F} and is generated by the global section $T: \mathcal{F} \rightarrow \mathcal{O}_V$ sending p to $\text{Tr}(p/J)$.

Remark. It is clear from the definition that the function $\text{Tr}(p/J)$ is an element of \mathcal{K}_V . Part of the content of the theorem is that this function is actually analytic.

References in place of a proof. This theorem is proved, for example, by Scheja and Storch in Sections 3 and 4 of [19] (see also Paragraph 2 and the first example on page 186 of [19]; note that the separability required on page 186 is automatic for us since $\text{char } \mathbf{R} = 0!$). Though Scheja-Storch treats only the case of algebras over rings, their proof extends without difficulty to the sheaf case above (see [12], II, Proposition 5, and [12], III for the necessary pieces of the theory of analytic sheaves). In fact, the result has a long history. An essentially equivalent theorem (with more restrictive hypotheses than the result of Scheja-Storch, but sufficiently general for our purposes) seems to have been proved first by Berger [5]. Berger proves his version by reducing to the 1-dimensional case, which was essentially well-known to number theorists in the last century—see [21], Chapter III, Proposition 14 for a very clear exposition of that case. The whole matter was vastly generalized by Grothendieck; his theory is thoroughly exposed in [11]. The result of Scheja-Storch is a more explicit version of a special case of Grothendieck's theory. A less complete, but more accessible introduction to Grothendieck's theory is given by Beauville [4], and a result of the explicitness we need can be derived from his treatment. Further references may be found in [19]. //

4D. How to compute the signature

We will retain the notations of sections 4B and 4C. The next proposition details the properties of the map $T: \mathcal{F} \rightarrow \mathcal{O}_V$ which make it useful to us.

PROPOSITION 4.4. *For any point $y \in V$,*

(i) $\text{Hom}_{\mathbf{R}}(\mathcal{Q}_y, \mathbf{R}) \cong \mathcal{Q}_y$ as a \mathcal{Q}_y -module; it is generated by the map T_y sending the class of a germ $p \in \mathcal{F}_y$ to $T(p)(y)$.

(ii) If e is any idempotent of \mathcal{F}_y , then $T(Je)(y) > 0$.

(iii) If \mathcal{N}_y is the intersection of the maximal ideals of \mathcal{F}_y , then $T(J\mathcal{N}_y)(y) = 0$.

Proof: (i): By Theorem 4.3, $T: \mathcal{F}_y \rightarrow \mathcal{O}_y$ generates $\text{Hom}_{\mathcal{O}_y}(\mathcal{F}_y, \mathcal{O}_y) \cong \mathcal{F}_y$. Since \mathcal{F}_y is a free \mathcal{O}_y -module, we have

$$\begin{aligned} \text{Hom}_{\mathcal{O}_y}(\mathcal{F}_y, \mathcal{O}_y) \otimes_y \mathcal{O}_y/\mathcal{N}_y &\cong \text{Hom}_{\mathcal{O}_y/\mathcal{N}_y}(\mathcal{F}_y/\mathcal{N}_y, \mathcal{O}_y/\mathcal{N}_y) \\ &= \text{Hom}(\mathcal{Q}_y, \mathbf{R}), \end{aligned}$$

so part (i) follows.

(ii): The definition of T , together with the idempotence of e , yields

$$\begin{aligned} T(Je) &= \text{Tr}(Je/J) \\ &= \text{Tr}(e) \\ &= r \cdot 1 \in \mathcal{O}_y \end{aligned}$$

where r is the rank of the free \mathcal{O}_y -module $e \cdot \mathcal{F}_y$, a positive integer.

(iii): Again, from the definition of T ,

$$T(J\mathcal{N}_y) = \text{Tr}(\mathcal{N}_y).$$

Now the evaluation at y is really reduction modulo \mathfrak{M}_y . Since \mathcal{F}_y is free over \mathcal{O}_y , and $\mathcal{N}_y \subset \mathcal{F}_y$ we can compute the reduction modulo \mathfrak{M}_y of the trace of an element of \mathcal{N}_y by first reducing the element modulo \mathfrak{M}_y , and then computing its trace as an endomorphism of $Q_y = \mathcal{F}_y/\mathfrak{M}_y$. That is,

$$\text{Tr}(\mathcal{N}_y)(y) = \text{Tr}(\mathcal{N}_y \cdot Q_y).$$

But $\mathcal{N}_y \cdot Q_y$ is the intersection of the maximal ideals of Q_y , so it consists of nilpotent elements. Thus $\text{Tr}(\mathcal{N}_y Q_y) = 0$. //

COROLLARY 4.5. *The residue class J_0 of J in $Q(f)$ generates the socle of $Q(f)$ and if $\varphi = Q(f) \rightarrow \mathbf{R}$ is any linear functional with $\varphi(J_0) > 0$, then the symmetric form*

$$\langle , \rangle_\varphi$$

of Theorem 1.2 is non-singular.

As mentioned in the beginning of this section, Corollary 4.5 is the last step in the deduction of Theorems 1.1 and 2.1 from Theorem 1.2.

Proof of Corollary 4.5. By Proposition 4.44(ii), the value of the functional T_0 on J_0 is positive; thus $J_0 \neq 0$. For the rest, note first that by Proposition 4.4(i), with $y = 0$ and Proposition 3.1, the theory of Section 3 applies to the \mathbf{R} -algebra $Q(f) = Q_0$. The fact that J_0 generates the socle, and, by Proposition 3.4, the nonsingularity of $\langle , \rangle_\varphi$, will follow if we show that J_0 is an element of the socle, that is, if we show that J_0 annihilates the maximal ideal \mathcal{N}_0 of Q_0 . But since $T_0: Q_0 \rightarrow \mathbf{R}$ generates Q_0^* , and since \mathcal{N}_0 is an ideal, it suffices by Corollary 3.2 to show that $\langle J, \mathcal{N}_0 \rangle_{T_0} = T_0(J\mathcal{N}_0) = 0$. This is an immediate consequence of Proposition 4.4(iii) with $y = 0$. //

The next result compares the forms $\langle , \rangle_\varphi$ and \langle , \rangle_{T_0} on $Q(f) = Q_0$, and \langle , \rangle_{T_y} on Q_y .

COROLLARY 4.6. *Let $\varphi: Q(f) \rightarrow \mathbf{R}$ be a linear functional with $\varphi(J_0) > 0$. For all $y \in V$, signature $\langle , \rangle_\varphi = \text{signature } \langle , \rangle_{T_y}$.*

Proof. Since the residue field of $Q(f)$ is \mathbf{R} , and positive numbers are squares in \mathbf{R} , Proposition 3.5(i) shows that $\langle , \rangle_\varphi$ and \langle , \rangle_{T_0} are equivalent. Thus it suffices to show the constancy of

signature \langle, \rangle_{T_y}

as a function of y .

By Proposition 4.4(i) and Proposition 3.1, \langle, \rangle_{T_y} is a nonsingular form on Q_y for every y . By Theorem 4.2(i), $\dim_{\mathbf{R}} Q_y = \text{rank}_{\mathcal{O}_y} \mathcal{F}_y$ is constant in y . We will show that \langle, \rangle_{T_y} is an analytically varying family of quadratic forms. Since they are all nonsingular of the same dimension, and since the signature is a discrete invariant, the signature must be constant.

It now suffices to exhibit matrices for the forms \langle, \rangle_{T_y} whose entries are analytic functions of y . Choose sections $a_1, \dots, a_m \in \mathcal{F}(V)$ which form a free basis of \mathcal{F} over \mathcal{O}_V . The residue classes of a_1, \dots, a_m form a vectorspace basis of Q_y for each y . The matrix of \langle, \rangle_{T_y} with respect to this basis has as its $(i, j)^{\text{th}}$ entry the number $T(a_i a_j)(y)$, which is by Theorem 4.3 an analytic function of y .

Proof of Theorem 1.2. It remains in view of Corollary 4.6, to prove that

$$\text{degree } f = \text{signature } \langle, \rangle_{T_y}$$

for some value of $y \in V$.

Since the regular values of f are dense, we may suppose that $y \in V$ is a regular value; that is, if x_1, \dots, x_k are the preimages of y contained in U , then with the notation of Theorem 4.2(iv),

$$(*) \quad Q_y = Q_y \bar{e}_1 \times \dots \times Q_y \bar{e}_k \times Q_y \bar{e}_{\text{complex}}$$

where \bar{e}_i denotes the image in $Q_y = \mathcal{F}_y / \mathcal{D}_y \mathcal{F}_y$ of e_i , $Q_y \bar{e}_i \cong \mathbf{R}$ for each i , and $Q_y \bar{e}_{\text{complex}}$ is a direct product of algebras whose residue class fields are isomorphic to \mathbf{C} .

To compute the signature of \langle, \rangle_{T_y} , note that since $\langle p, q \rangle_{T_y}$ is computed by first multiplying p and q , the decomposition in $(*)$ is an orthogonal decomposition of the form. Thus we may compute the signature of \langle, \rangle_{T_y} one factor at a time. Furthermore, by Proposition 3.1 each direct factor of Q_y together with the restriction of T_y , satisfies the conditions of Section 3. Since the residue class fields of $Q_y \bar{e}_{\text{complex}}$ are complex, Corollary 3.7 shows that the local direct factors of $Q_y \bar{e}_{\text{complex}}$ contribute nothing to the signature.

It remains to compute the signature on a factor of the form $Q_y \bar{e}_i$ corresponding to a point $x_i \in f^{-1}(y)$. Since y is a regular value of f , the value $J(x_i)$ of the Jacobian determinant J at x_i is nonzero. The image $J\bar{e}_i$ of J in $Q_y \bar{e}_i$ is $J(x_i) \cdot \bar{e}_i$ and thus forms a basis of $Q_y \bar{e}_i$. The signature of the restriction of \langle, \rangle_{T_y} to $Q_y \bar{e}_i$ is plus or minus 1 according to whether $\langle J\bar{e}_i, J\bar{e}_i \rangle$ is positive or negative, but

$$\begin{aligned}\langle J\bar{e}_i, J\bar{e}_i \rangle &= T_y(J^2\bar{e}_i^2) \\ &= J(x_i)(T_y J\bar{e}_i) \\ &= J(x_i)[T(Je_i)(y)],\end{aligned}$$

which has the same sign as $J(x_i)$ by Proposition 4.4(ii).

Thus

$$\text{signature } \langle , \rangle_{T_y} = \sum_{x \in f^{-1}(y) \cap U} \text{sign } J(x) = \text{deg } f. \quad //$$

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Appendix

Sur une inégalité à la Minkowski pour les multiplicités

By B. TEISSIER

Introduction

Soit \mathcal{O} un anneau local noetherien (on pourra penser par exemple à une \mathbb{C} -algèbre analytique, ou formelle, c'est-à-dire à un quotient d'une algèbre de séries convergentes ou formelles en un nombre fini de variables), et soient n_1, \dots, n_l des idéaux de \mathcal{O} primaires pour l'idéal maximal m , autrement dit, tels qu'il existe des entiers α_r tels que $m^{\alpha_r} \subset n_r \subset m$ ($1 \leq r \leq l$). J.J. Risler et l'auteur ont démontré (voir [T], Chap. 1, § 2) que l'application $H: \mathbb{N}^l \rightarrow \mathbb{N}$ définie par

$$H(\nu_1, \dots, \nu_l) = \lg_{\mathcal{O}} \mathcal{O}/n_1^{\nu_1} \cdots n_l^{\nu_l}$$

(ce qui donne $H(\nu_1, \dots, \nu_l) = \dim_{\mathbb{C}} \mathcal{O}/n_1^{\nu_1} \cdots n_l^{\nu_l}$ dans le cas d'une algèbre analytique) prend, quand les ν_i sont assez grands, les mêmes valeurs qu'un polynôme en les ν_i à coefficients rationnels, de degré $d = \dim \mathcal{O}$. De plus, les termes de plus haut degré de ce polynôme s'écrivent:

$$\bar{H}(\nu_1, \dots, \nu_l) = \frac{1}{d!} \sum_{\substack{\alpha \in \mathbb{N}^l \\ |\alpha| = d}} \frac{d!}{\alpha!} e(n^{[\alpha]}) \nu_1^{\alpha_1} \cdots \nu_l^{\alpha_l}$$

où $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l$, $|\alpha| = \sum \alpha_i$, $\alpha! = \alpha_1! \cdots \alpha_l!$ et où $n^{[\alpha]}$ désigne, comme par la suite, un idéal engendré par α_i combinaisons linéaires "assez générales" d'un système de générateurs de l'idéal n . Dans le cas où \mathcal{O} est une algèbre analytique, $n^{[\alpha]}$ désignera si l'on veut l'idéal engendré par α_i combinaisons linéaires assez générales à coefficients dans \mathbb{C} . Cette notation incite à écrire des égalités telles que $n^{[1]} + n^{[1]} = n^{[2]}$, ce que je me permettrai de faire plus bas. Enfin, la notation $e(n)$ désigne classiquement la multiplicité au sens de Samuel de l'idéal primaire n , qui est définie par l'égalité:

$$\lg_{\mathcal{O}} \mathcal{O}/n^{\nu} = \frac{e(n)}{d!} \nu^d + \mathbf{0}(\nu^{d-1}) \quad (e(n) \in \mathbb{N})$$

(c'est le cas $l = 1$ du résultat ci-dessus, dont l'on tire en même temps

$e(n) = e(n^{[d]})$, égalité due à Samuel (cf. [Se]).

Tout cela nous fournit une expression pour la multiplicité du produit de deux idéaux primaires; en prenant $l = 2$, et $\nu_1 = \nu_2$, il vient:

$$(E.1): \quad e(n_1 \cdot n_2) = \sum_{i=0}^d \binom{d}{i} e(n_1^{[i]} + n_2^{[d-i]}).$$

Cette formule du binôme symbolique pour les $(e(n))^{1/d}$ m'avait conduit ([T], Chap. I, § 2) à conjecturer l'inégalité "à la Minkowski":

$$(I.1): \quad e(n_1 \cdot n_2)^{1/d} \leq e(n_1)^{1/d} + e(n_2)^{1/d}$$

et en fait les inégalités

$$(I.2): \quad e(n_1^{[i]} + n_2^{[d-i]})^d \leq e(n_1)^i \cdot e(n_2)^{d-i} \quad (0 \leq i \leq d)$$

qui, au vu de (E.1), entraînent [I.1]. Remarquons que pour $d = 0$, $e(n_1 \cdot n_2) = e(n_1) = e(n_2)$, et pour $d = 1$, [I.2] est clair car sûrement $e(n_i) \geq 1$. En fait, quand $d = 1$, $e(n_1 \cdot n_2) = e(n_1) + e(n_2)$.

Je vais démontrer ici [I.2], et donc [I.1], dans le cas où \mathcal{O} est une algèbre noetherienne réduite de Cohen-Macaulay sur un corps algébriquement clos de caractéristique zéro (ou plus généralement, sur lequel on dispose de la résolution des singularités des surfaces).

Dans ce cas, on a en fait ([Se]), par la propriété de Cohen-Macaulay:

$$(E.2): \quad e(n_1^{[i]} + n_2^{[d-i]}) = \lg_{\mathcal{O}} \mathcal{O}/n_1^{[i]} + n_2^{[d-i]} = \dim_k \mathcal{O}/n_1^{[i]} + n_2^{[d-i]}$$

car $n_1^{[i]} + n_2^{[d-i]}$ est nécessairement engendré par une suite régulière. J'utiliserai ceci, mais garderai cependant la notation de multiplicité de préférence à celle de longueur, car la multiplicité est invariante par des opérations sur les idéaux qui détruisent la propriété d'être engendré par une suite régulière, par exemple la fermeture intégrale. Par ailleurs, si \mathcal{O} est de Cohen-Macaulay, son complété m -adique $\hat{\mathcal{O}}$ l'est aussi, et l'on a $e(n) = e(n \cdot \hat{\mathcal{O}})$. On peut donc supposer \mathcal{O} complet, et donc que l'on sait résoudre les singularités des surfaces correspondant aux quotients de dimension 2 de \mathcal{O} .

Exemples d'applications

A) Prenons $\mathcal{O} = \mathbb{C}\{X_1 \dots X_n\}$ et

$$n_1 = (f_1, \dots, f_n), \quad n_2 = (g_1, \dots, g_n), \quad f_i, g_j \in \mathcal{O}.$$

Géométriquement, nous nous donnons deux germes de morphismes finis $f, g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ et le degré de f (resp. g) vaut $\dim_{\mathbb{C}} \mathcal{O}/n_1 = e(n_1)$ (resp. $\dim_{\mathbb{C}} \mathcal{O}/n_2 = e(n_2)$). On peut se demander quel est le degré du germe de morphisme $f^{[i]}, g^{[n-i]}: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ donné par $(f_1, \dots, f_i, g_{i+1}, \dots, g_n)$, au moins dans le cas où f_1, \dots, f_n et g_1, \dots, g_n sont des générateurs assez

généraux de n_1 et n_2 respectivement. Bien sûr, $\deg(\underline{f}^{[i]}, \underline{g}^{[n-i]}) = e(n_1^{[i]} + n_2^{[n-i]})$ dans ce cas et donc [I.2] nous fournit l'inégalité:

$$\deg(\underline{f}^{[i]}, \underline{g}^{[n-i]}) \leq (\deg \underline{f})^{\frac{i}{n}} \cdot (\deg \underline{g})^{\frac{n-i}{n}}.$$

En particulier, si g est l'identité, c'est-à-dire $n_2 = (X_1, \dots, X_n) = m$, idéal maximal de \mathcal{O} , on a $\deg g = 1$ et il vient $\deg(\underline{f}^{[i]}, \text{id.}^{[n-i]}) \leq (\deg \underline{f})^{i/n}$ et plus particulièrement, si nous prenons $i = n - 1$, nous pouvons remarquer que $\deg(\underline{f}^{[n-1]}, \text{id.}^{[1]}) = m_0(\Gamma)$, multiplicité à l'origine de la courbe Γ de \mathbb{C}^n définie par $f_1 = \dots = f_{n-1} = 0$, disons (ou $n - 1$ combinaisons génériques des générateurs de n_1), et que d'après [I.2]:

$$\deg(\underline{f}^{[n-1]}, \text{id.}^{[1]}) = m_0(\Gamma) \leq (\deg \underline{f})^{1-\frac{1}{n}}.$$

En effet, $m_0(\Gamma) = e(m \cdot \mathcal{O}_\Gamma) = e(m^{[1]} \cdot \mathcal{O}_\Gamma)$ puisque $\dim \Gamma = 1$, et \mathcal{O}_Γ est de Cohen-Macaulay, étant intersection complète.

B) Prenons maintenant $\mathcal{O} = \mathcal{O}_{S_0,0}$ algèbre d'un germe de surface analytique complexe S_0 de Cohen-Macaulay réduite, et $n_1 = (y_1, y_2)$, $n_2 = m$, idéal maximal. Supposons y_i non diviseur de 0, et notons C_i la courbe sur S_0 définie par $y_i = 0$. Par définition, la multiplicité d'intersection en 0, $(C_1, C_2)_0$, des deux courbes C_1 et C_2 sur S_0 est $e((y_1, y_2) \cdot \mathcal{O}_{S_0,0})$, et la *méthode de démonstration* de (I.2) (2.1 ci-dessous) nous fournirait dans ce cas:

$$(C_1, C_2)_0 \geq \frac{m_0(C_1) \cdot m_0(C_2)}{m_0(S_0)}$$

(inégalité bien connue dans le cas où S_0 est non singulière) où m_0 désigne toujours la multiplicité à l'origine.

En définissant comme Mumford [M] la multiplicité d'intersection de deux diviseurs de Weil sur S_0 , l'argument que je donne au Paragraphe 2 ci-dessous permet d'étendre cette inégalité au cas où C_1 et C_2 sont des diviseurs de Weil effectifs (c'est-à-dire des combinaisons à coefficients entiers positifs de courbes irréductibles sur S_0) et non plus de Cartier, la multiplicité d'intersection pouvant alors être un nombre rationnel non entier. Par exemple, la multiplicité d'intersection au sommet de deux génératrices d'un cône quadratique vaut $1/2$.

C) Prenons $\mathcal{O} = \mathbb{C}\{z_0, \dots, z_n\}$, et soit $f \in \mathcal{O}$ un générateur pour l'idéal d'un germe d'hypersurface à singularité isolée $(X_0, 0) \subset (\mathbb{C}^{n+1}, 0)$. Prenant $n_1 = j(f) = (\partial f / \partial z_0, \dots, \partial f / \partial z_n) \cdot \mathcal{O}$ et $n_2 = m = (z_0, \dots, z_n) \cdot \mathcal{O}$, l'inégalité [I.3] que nous allons démontrer ci-dessous devient (cf. [T], Chap. I.)

$$\frac{\mu^{(n+1)}}{\mu^{(n)}} \geq \frac{\mu^{(n)}}{\mu^{(n-1)}} \geq \dots \geq \frac{\mu^{(i)}}{\mu^{(i-1)}} \geq \dots \geq \frac{\mu^{(1)}}{\mu^{(0)}}$$

où $\mu^{(i)}$ est le nombre de Milnor de l'intersection de X_0 avec un i -plan général de \mathbb{C}^{n+1} passant par 0. Ceci répond à la question de [T], Ch. II, 2.2.

1. La méthode de démonstration de [I.2] consiste à prouver un système d'inégalités à priori plus fort, pour la preuve duquel on peut se ramener à démontrer [I.2] dans le cas $d = 2$, et ensuite à vérifier que dans ce cas l'inégalité cherchée équivaut essentiellement au fait que la matrice d'intersection des composantes du diviseur exceptionnel d'une résolution des singularités d'un germe de surface normale est définie négative.

Voici le système d'inégalités en question: Posons pour simplifier l'écriture $e_i = e(n_1^{[i]} + n_2^{[d-i]})$. Je dis que l'on a

$$(I.3): \quad \frac{e_i}{e_{i-1}} \geq \frac{e_{i-1}}{e_{i-2}} \quad (2 \leq i \leq d).$$

1.1. Commençons par vérifier (I.3) \Rightarrow (I.2). Pour cela remarquons (E.2) que:

$$e_i = e((n_1^{[i]} + n_2^{[d-j-i]}) \cdot \mathcal{O}/n_2^{[j]}) \quad \text{pour } 0 \leq i \leq d - j$$

et $\dim \mathcal{O}/n_2^{[j]} = d - j$. Choisissons $j = 1$, et posons $\tilde{n}_1 = n_1 \cdot \mathcal{O}/n_2^{[1]}$, $\tilde{n}_2 = n_2 \cdot \mathcal{O}/n_2^{[1]}$ et donc pour $0 \leq i \leq d - 1$, $e_i = e(\tilde{n}_1^{[i]} + \tilde{n}_2^{[d-1-i]})$, et ainsi, nous pouvons, raisonnant par récurrence sur d , supposer [I.2] vérifiée jusqu'en dimension $d - 1$.

C'est-à-dire supposer que nous avons:

$$e_i^j \leq e_j^i \cdot e_0^{j-i} \quad (0 \leq i \leq j \leq d - 1)$$

et utiliser (I.3) pour prouver l'inégalité de (I.2) qui nous manque, c'est-à-dire $e_i^d \leq e_d^i \cdot e_0^{d-i}$. Cette dernière inégalité est clairement vérifiée par $i = 0, d$ et nous supposons donc $1 \leq i \leq d - 1$. Or, élevons les deux membres de (I.3) prise pour $i = d$ à la puissance i ,

$$\left(\frac{e_d}{e_{d-1}} \right)^i \geq \left(\frac{e_{d-1}}{e_{d-2}} \right)^i$$

et utilisons notre hypothèse de récurrence sous la forme:

$$e_{d-1}^i \geq \frac{e_i^{d-1}}{e_0^{d-1-i}}$$

ce qui nous donne:

$$e_d^i \cdot e_0^{d-i} \geq e_i^{d-1} \cdot e_0 \cdot \left(\frac{e_{d-1}}{e_{d-2}} \right)^i.$$

Il nous suffit donc pour conclure de prouver:

$$e_0 \cdot \left(\frac{e_{d-1}}{e_{d-2}} \right)^i \geq e_i$$

ou encore $(e_{d-1}/e_{d-2})^i \geq e_i/e_0$. Or, (I.3) nous permet d'écrire

$$\frac{e_{d-1}}{e_{d-2}} \geq \dots \geq \frac{e_i}{e_{i-1}} \geq \frac{e_{i-1}}{e_{i-2}} \geq \dots \geq \frac{e_1}{e_0}$$

et donc

$$\left(\frac{e_{d-1}}{e_{d-2}}\right)^i \geq \frac{e_i}{e_{i-1}} \cdot \frac{e_{i-1}}{e_{i-2}} \cdot \dots \cdot \frac{e_1}{e_0} = \frac{e_i}{e_0},$$

ce qui achève de prouver (I.3) \Rightarrow (I.2).

2. Venons-en maintenant à la démonstration de (I.3). À nouveau par la remarque faite en 1.1, nous voyons par récurrence sur d qu'il suffit de démontrer:

$$\frac{e_d}{e_{d-1}} \geq \frac{e_{d-1}}{e_{d-2}}$$

et que, posant $\tilde{n}_1 = n_1 \cdot \mathcal{O}/n_1^{[d-2]}$ et $\tilde{n}_2 = n_2 \cdot \mathcal{O}/n_1^{[d-2]}$,

$$e_d = e(\tilde{n}_1^{[2]}) = \tilde{e}_2, \quad e_{d-1} = e(\tilde{n}_1^{[1]} + \tilde{n}_2^{[1]}) = \tilde{e}_1 \quad \text{et} \quad e_{d-2} = e(\tilde{n}_2^{[2]}) = \tilde{e}_0.$$

De plus, $\mathcal{O}/n_1^{[d-2]}$ est l'algèbre d'un germe de surface de Cohen Macaulay réduite (grâce au théorème de Bertini), que je noterai S_0 . Démontrer $e_d/e_{d-1} \geq e_{d-1}/e_{d-2}$ revient donc à montrer $\tilde{e}_2 \cdot \tilde{e}_0 \geq \tilde{e}_1^2$, c'est-à-dire que nous avons ramené la démonstration de (I.3) en général à celle de (I.2) quand $d = 2$.

2.1. Démonstration de (I.2) dans le cas $d = 2$ (\tilde{n}_i est écrit n_i). Après le travail de Hironaka [H] pour k de caractéristique 0 (ou celui de Abhyankar [Ab] pour une algèbre locale sur un corps k de caractéristique quelconque) nous pouvons prendre une résolution des singularités du germe de surface S_0 correspondant à \mathcal{O} , c'est-à-dire un morphisme propre $\pi: S \rightarrow S_0$ induisant un isomorphisme en dessus d'un ouvert dense de S_0 (et donc surjectif) et tel que

- 1) S soit une surface non singulière;
- 2) $n_1^{[2]} \cdot \mathcal{O}_S$, $(n_1^{[1]} + n_2^{[1]}) \cdot \mathcal{O}_S$ et $n_2^{[2]} \cdot \mathcal{O}_S$ soient tous des idéaux inversibles de \mathcal{O}_S , ainsi que $m \cdot \mathcal{O}_S$ où m est l'idéal maximal de $\mathcal{O}_{S_0,0} = \mathcal{O}$;
- 3) $(\pi^{-1}(0))_{\text{red}} = \bigcup_{k=1}^2 E_k$ soit un diviseur à croisements normaux de S , chaque E_k étant une courbe lisse;
- 4) Si nous écrivons $n_1^{[2]} = (y_1, y_2)\mathcal{O}_{S_0,0}$, $n_2^{[2]} = (z_1, z_2)\mathcal{O}_{S_0,0}$ et $n_1^{[1]} + n_2^{[1]} = (y_1, z_1)\mathcal{O}_{S_0,0}$ (ou $(y_1, z_2)\mathcal{O}_{S_0,0}$ ou $(y_2, z_2)\mathcal{O}_{S_0,0}$), les transformées strictes des courbes de S_0 définies par $y_i = 0$ ou $z_j = 0$ sont disjointes deux à deux. (Pour obtenir une telle résolution, on peut commencer par éclater le produit $m \cdot n_1^{[2]} \cdot n_2^{[2]} \cdot (n_1^{[1]} + n_2^{[1]})$ puis résoudre les singularités du résultat de façon à satisfaire 3) et 4.)

2.2. Faits: Dans cette situation on a :

1) Pour toute fonction $h \in \mathcal{O}_{S_0,0}$ non diviseur de 0, si nous notons $A(h)$ le diviseur de Cartier de S défini par $\pi \circ h$, i.e. $h \cdot \mathcal{O}_S$, nous avons :

$$\langle A(h), E_k \rangle = 0 \quad 1 \leq k \leq t$$

où $\langle \rangle$ désigne le nombre d'intersection des diviseurs sur S .

En effet, par conservation du nombre d'intersection par équivalence linéaire [S], si nous regardons $S_0 \times \mathbb{C}$, nous avons

$$\langle A(h), E_k \rangle = \langle A(h+t), E_k \rangle \text{ et } A(h+t) \cap E_k = \emptyset \text{ car } E_k \subset \pi^{-1}(0).$$

2) La matrice d'intersection ($\langle E_k, E_{k'} \rangle$) est négative. On peut en effet décomposer $\pi^{-1}(0) = \bigcup E_k$ en ses composantes connexes (correspondant aux composantes analytiquement irréductibles de $(S_0, 0)$) et appliquer à chacune de celles-ci l'argument donné dans Mumford [M], qui fonctionne même dans le cas des surfaces non normales.

3) Le diviseur A_i (resp. B_j) de S défini par $y_i \cdot \mathcal{O}_S$ (resp. $z_j \cdot \mathcal{O}_S$), peut s'écrire (comme cycle)

$$A_i = A'_i + \sum v_k(y_i) \cdot E_k \\ (\text{resp. } B_j = B'_j + \sum v_k(z_j) \cdot E_k)$$

où A'_i (resp. B'_j) désigne la courbe transformée stricte par π de la courbe sur S_0 définie par $y_i = 0$ (resp. $z_j = 0$), et, pour toute fonction $h \in \mathcal{O}_{S_0,0}$, $v_k(h)$ désigne l'ordre avec lequel $h \circ \pi = h \cdot \mathcal{O}_S$ s'annule le long de E_k au voisinage d'un point général de E_k (cf. [S]).

4) On a une "formule de projection biméromorphe" (cf. [R])

$$(e(n_1) =)e(n_1^{[2]}) = e(y_1, y_2)_{\mathcal{O}_{S_0,0}} = \langle A'_1 + \sum v_k(y_1)E_k, A'_2 + \sum v_k(y_2)E_k \rangle$$

et de même pour les autres $e((n_1^{[i]} + n_2^{[j]})_{\mathcal{O}})$. En appliquant 4) à $n_1^{[2]}$, $n_2^{[2]}$ et $n_1^{[1]} + n_2^{[1]}$, ce que nous avons à démontrer est en fait :

$$\langle A'_1 + \sum v_k(y_1)E_k, A'_2 + \sum v_k(y_2)E_k \rangle \langle B'_1 + \sum v_k(z_1)E_k, B'_2 + \sum v_k(z_2)E_k \rangle \geq \\ \langle A'_1 + \sum v_k(y_1)E_k, B'_1 + \sum v_k(z_1)E_k \rangle \langle A'_2 + \sum v_k(y_2)E_k, B'_2 + \sum v_k(z_2)E_k \rangle.$$

Notons tout de suite qu'en vertu du caractère général des y_i et z_j , nous avons

$$v_k(y_1) = v_k(y_2) \text{ et } v_k(z_1) = v_k(z_2). \quad (1 \leq k \leq t).$$

Par ailleurs, en utilisant la linéarité des multiplicités d'intersection, et le fait que par exemple

$$\langle A'_1 + \sum v_k(y_1)E_k, E_{k'} \rangle = 0 \quad (1 \leq k' \leq t)$$

d'après 2.2,1, et les analogues pour les autres on peut écrire par exemple :

$$\langle A'_1 + \sum v_k(y_1)E_k, A'_2 + \sum v_k(y_2)E_k \rangle = \langle A'_1, A'_2 + \sum v_k(y_2)E_k \rangle$$

et puisque les transformées strictes sont disjointes, ceci vaut $\langle A'_1, \sum v_k(y_2)E_k \rangle$ qui est, en appliquant à nouveau 2.2,1, $-\langle \sum v_k(y_1)E_k, \sum v_k(y_2)E_k \rangle$.

En appliquant cet argument à toutes nos multiplicités d'intersection sur S , notre inégalité à démontrer devient, en posant $v_k(y_1) = v_k(y_2) = u_k$, $v_k(z_1) = v_k(z_2) = v_k$

$$\left(\sum \langle E_k, E_{k'} \rangle u_k \cdot u_{k'}\right) \left(\sum \langle E_k, E_{k'} \rangle v_k \cdot v_{k'}\right) \geq \left(\sum \langle E_k, E_{k'} \rangle u_k \cdot v_{k'}\right)^2.$$

Ainsi, en notant U (resp. V) le vecteur des u_k (resp. v_k), et Φ la forme bilinéaire associée à la matrice $(\langle E_k, E_{k'} \rangle)$ nous voulons montrer que $\Phi(U, U) \cdot \Phi(V, V) \geq \Phi(U, V)^2$ sachant que $\Phi(U + \lambda V, U + \lambda V) \leq 0$ pour tout $\lambda \in \mathbf{R}$ puisque Φ est négative (2.2.,2). Ainsi, $\Phi(U + \lambda V, U + \lambda V) = \Phi(U, U) + 2\lambda\Phi(U, V) + \lambda^2\Phi(V, V)$ ne change pas de signe ($\lambda \in \mathbf{R}$), et donc son discriminant

$$\Delta = \Phi(U, V)^2 - \Phi(U, U) \cdot \Phi(V, V) \leq 0$$

ce qui achève de prouver l'inégalité (I.3) et donc les deux autres.

Question: Supposant \mathcal{O} normal, est-il vrai que l'on a l'égalité $e(n_1 \cdot n_2)^{1/d} = e(n_1)^{1/d} + e(n_2)^{1/d}$ si et seulement si il existe des entiers a et b tels que n_1^a et n_2^b aient même fermeture intégrale? (added in proof) La réponse à cette question est affirmative, comme il est montré dans "sur une inégalité à la Minkowski II", à paraître.

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