NOTE ON THE TOPOLOGICAL DEGREE OF A SMOOTH MAPPING*

by

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One of the fundamental tools for the analysis of a continuous map

$$f: M^n \to N^n$$

of compact, connected, oriented n-dimensional manifolds is the <u>topological degree</u> of f, which may be defined as the image of $1 \in Z$ under the homomorphism

$$Z \approx H_n(M) \xrightarrow{f_*} H_n(N) \approx Z \quad ,$$

where Z denotes the ring of integers, and f_* is the map induced by f on homology.

From this definition it is clear that <u>deg f</u>, the degree of f, really is a topological (even a homotopy-theoretic) invariant of f. However, if M, N, and f are all <u>smooth</u> (that is, infinitely differentiable), the degree takes on new meanings. (A very beautiful and elementary exposition of some examples of this may be found in [M-1].) In this note we will discuss an algebraic interpretation of the degree and one of its geometric consequences.

We will assume from now on that M, N and f are smooth. In this case, one can describe $\deg f$ in terms of the "local" behaviour of f. Recall, first, that a point $x \in M$ is said to be a <u>regular point</u> for f if the map

$$df_x: T_xM \to T_{f(x)}N \quad ,$$

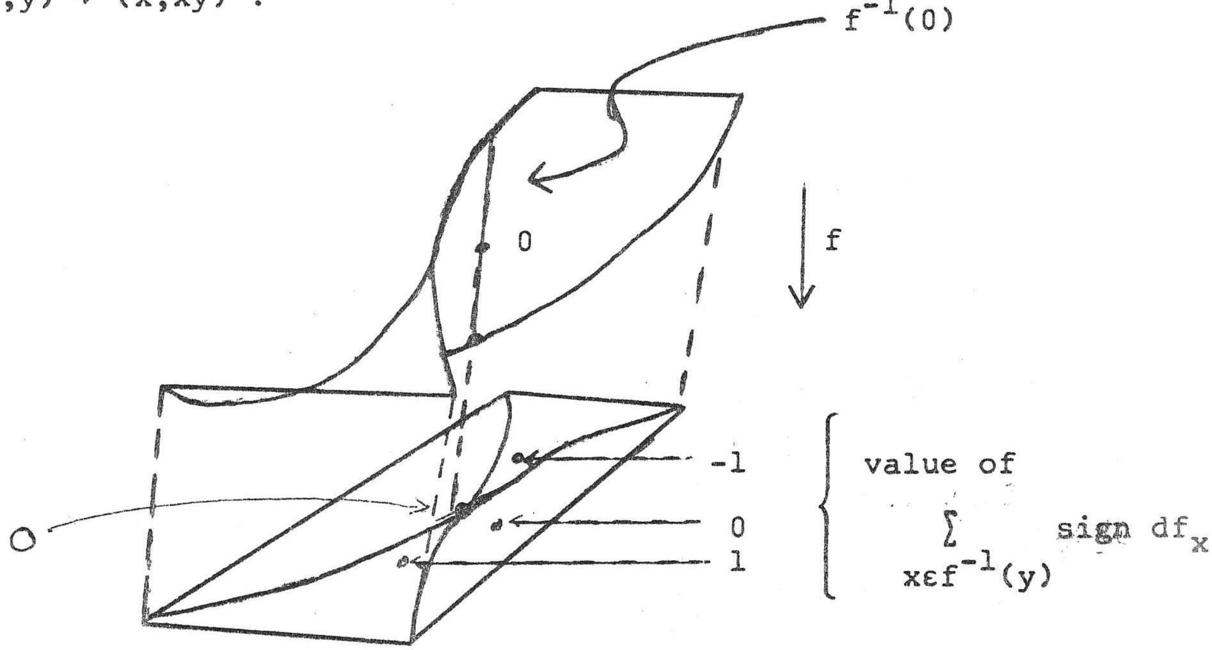
between the tangent spaces to M and N, induced by f, is nonsingular. A point $y \varepsilon N$ is a <u>regular value</u> for f if $f^{-1}(y)$ consists of regular points (or is empty). It follows that $f^{-1}(y)$ is a finite set. Since M and N are oriented, so are $T_x M$ and $T_{f(x)} N$, and we define the sign of df_x to be 1 or -1, depending on whether df_x preserves or reverses this orientation. For any regular value y of f one then has:

$$\deg f = \sum_{x \varepsilon f^{-1}(y)} \operatorname{sign} df_x \quad .$$

(See [M1; p.27]). If we define the <u>local degree</u> $\deg_x f$ of f at a regular point $x \varepsilon M$ to be sign df_x, this becomes

$$(*) \qquad \deg f = \sum_{x \varepsilon f^{-1}(y)} \deg_x f \quad .$$

This notion of local degree can be extended to singular points as well, using formula (*) as a guide. In general, if $f_0 : (\mathbb{R}^n, 0) \dashrightarrow (\mathbb{R}^n, 0)$ is the <u>germ</u> of a smooth map (we will use broken arrows to denote germs, and we will usually abbreviate the above expression to $f_0 : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$), defined on some neighborhood U of 0, we again wish to define the degree of f_0 by (*), where now y must be chosen to be a regular value sufficiently near 0. To make this independent of the regular value chosen, however, we need to have some strong finiteness condition on f . For example, if f_0 is the germ of the map from $(0,1)\times(0,1)$ to $(0,1)\times(0,1)$ given by $(x,y) \to (x,xy)$:



then (*) yields -1 , 0 ,or 1 at points y in <u>any</u> neighborhood of 0 ! The "best" condition to choose turns out to be the following: we will say that $f_0 : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ is <u>finite</u> at 0 if the coefficients $f^1,\ldots,f^n$ of f_0 generate an ideal (f) in the ring $C = C^\infty_{0,0}(\mathbb{R}^n)$ of germs of smooth germs of functions $\mathbb{R}^n \dashrightarrow \mathbb{R}$ carrying 0 to 0 such that the factor ring

$$Q(f) = C/(f)$$

is finite dimensional over $\mathbb{R}$.

Under the assumption that f_0 is finite, we may define

$$\deg_0 f = \deg f_0 = \sum_{x \varepsilon f_0^{-1}(y)} \deg_x f_0 \quad ,$$

for any regular value y of f_0 sufficiently near 0.

Henceforward, then, let $f: R^n \longrightarrow R^n$ be a finite germ with $f(0) = 0$. We propose to describe the degree of f <u>without</u> "moving off to a regular value" y, in terms of the finite dimensional $\mathbb{R}$-algebra $Q(f) = C/(f)$. Of course, no choice of orientation went into the definition of $Q(f)$, so the best one can hope to get out of just the algebra structure of $Q(f)$ is the absolute value of the degree.

<u>Theorem 1</u>: Let I be an ideal of $Q(f)$, maximal with respect to the property $I^2 = 0$. Then

$$|\deg f| = (\dim_R Q(f)) - (2 \dim_R I) \quad .$$

<u>Corollary 1</u>: If f is not regular at 0, then $\deg f < \dim Q(f)$.

This follows because any artinian local ring which is not a field has a nonzero ideal with square 0.

The Corollary has the following geometric interpretation: Let $\tilde{f}$ be a polynomial map which agrees with f up to high order, and let $\tilde{f}_{\mathbb{C}}: \mathbb{C}^n \to \mathbb{C}^n$ be the map given by the same polynomials as $\tilde{f}$. Then $\dim Q(f)$ can be identified with the number of points near 0 in $\tilde{f}_{\mathbb{C}}^{-1}(y)$, where y is a regular value for $\tilde{f}_{\mathbb{C}}$ near 0. (See for example [M2] appendix B). Such a regular value y can always be chosen to have real coordinates, and it may happen that the points of $\tilde{f}^{-1}(y)$ also have real coordinates; we will describe this situation by saying that "the complex preimages of y under f are all real". (This may depend on the choice of $\tilde{f}$!) With this language we can state the geometric content of the Corollary as follows:

Corollary 2: Suppose 0 is a singular point of the finite map germ f. Then for each point $y \varepsilon R^n$ near 0, either

i) for every choice of $\tilde{f}$, not all the complex preimages of y are real

or

ii) f is orientation preserving at some point of $f^{-1}(y)$ and orientation reversing at another point.

To get hold of the degree itself we need slightly more data. Let $J = \det(\frac{\partial f}{\partial x})$ be the jacobian of f. It is a nonzero smooth germ on R^n, and as such has a residue class $J+(f)$ in $Q(f) = C/(f)$, which we also call J.

Recall that the signature of a symmetric bilinear form over R is the number of positive eigenvalues minus the number of negative eigenvalues of a matrix for the form.

Theorem 2:

1) $J \neq 0$ in $Q(f)$

2) If $\phi: Q(f) \to R$ is any R-linear functional such that $\phi(J) > 0$, then the symmetric bilinear defined by

$$\langle p,q \rangle_\phi = \phi(pq) \quad \text{for } p,q \in Q(f)$$

is nonsingular, and

3) deg f = signature $\langle\ ,\ \rangle$.

A proof of Theorem 2 is outlined in [E-L1], and will presumably be available, with details, in [E-L2]. Rather than discussing it here, we will show how Theorem 1 follows from Theorem 2.

First we need some remarks on bilinear forms. If $\langle\ ,\ \rangle$ is a symmetric bilinear form on a vectorspace V over a field k of characteristic $\neq 2$, then a subsapce H of V is said to be isotropic if $\langle H,H \rangle = 0$. Over R every nonsingular form decomposes into an orthogonal sum of a <u>definite</u> part (with no isotropic subspace), and a <u>hyperbolic</u> part, whose dimension is twice the dimension of a maximal isotropic subspace. The dimensions of these two parts are uniquely determined; the dimension of the definite part is the absolute value of the signature. Theorem 1 thus follows immediately from Theorem 2 and the next proposition.

<u>Proposition 3</u>: Let k be a field, and let A be a finite dimensional commutative local k-algebra with residue class field k. Let $\phi: A \to k$ be a k-linear functional, and let $\langle\ ,\ \rangle$ be a symmetric bilinear form on A defined by the formula $\langle p,q\rangle = \phi(pq)$. Suppose $\langle\ ,\ \rangle$ is nonsingular, and let I be an ideal of A which is maximal with respect to the condition $I^2 = 0$. Then I is a maximal isotropic subspace of A.

<u>Proof</u>: If $I^2 = 0$, then $\langle I,I\rangle = \phi(I^2) = 0$, so I is isotropic; suppose it were not maximal among isotropic subspaces. Let $p \in A$ be an element such that $p \notin I$, but $kp \oplus I$ is an isotropic subspace. Let M be the maximal ideal of A, and choose $a \in A$ such that $ap \notin I$ but $Map \subseteq I$. (If $Mp \subset I$ already, choose $a=1$). We will show that

$$(*) \qquad (I+Aap)^2 = 0 ,$$

contradicting the maximality of I.

First of all, $I \oplus kap$ is an isotropic subspace. For,

$\langle I,I\rangle = \phi(I^2) = 0$ since I is isotropic;

$\langle I,ap\rangle = \phi(Iap) \subset \phi(Ip) = \langle I,p\rangle = 0$

since I is an ideal and $I \oplus kp$ is isotropic;

if $a \in M$, then

$$\langle ap,ap\rangle = \phi(a^2p^2) = \langle a^2p,p\rangle \subseteq \langle Map,p\rangle \subseteq \langle I,p\rangle = 0 ,$$

while if $a \notin M$, then, by choice, a=1, and

$$\langle ap,ap\rangle = \langle p,p\rangle = 0$$, since $I \oplus kp$ is isotropic.

Moreover, $I \oplus kap = I + Aap$, since $A = k \oplus M$, and $Map \subseteq I$. Thus it suffices to prove that if J is an isotropic ideal, then $J^2 = 0$. But $0 = \langle J,J\rangle = \phi(J^2) = \langle A,J^2\rangle$. Since $\langle\ ,\ \rangle$ is nonsingular, $J^2 = 0$.

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