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SOME DIRECTIONS OF RECENT PROGRESS IN COMMUTATIVE ALGEBRA

David Eisenbud*

ABSTRACT

Three recently active areas of commutative algebra are discussed, and some results from each are presented. The areas are

1) Projective modules over polynomial rings.
2) The recent work on the existence of Cohen-Macaulay modules, and its relation to the conjectures on the rigidity of Tor, and on multiplicities.
3) Ideals of low codimension; two applications of the structure theorem for perfect ideals of codimension 2.

This article contains the write-ups of three independent talks on areas of commutative algebra which have shown what seems to me striking recent progress. They are also areas which ought to go on developing -- nearly all the main problems are still unsolved.

I have not tried to merge the three talks; each even retains its own references.

I. THE SERRE PROBLEM ON PROJECTIVE MODULES

The problem, posed by Serre in 1954 [4], is: Let k be a field; is every projective $k[x_1,\dots,x_n]$-module free? Equivalently, is every algebraic vector-bundle on affine n-space over k free?

Progress on this question was smooth, if slow, until the early sixties, thanks to the work of Serre, Seshadri, and Bass. By that time the answer to the question itself was known only for $n \le 2$ ("Seshadri's theorem" - the answer is "yes" in this case); but there was a wealth of subsidiary information on stable freeness and cancellation, which showed, for instance, that projective $k[x_1,\dots,x_n]$-modules of rank $\ge n+1$ are free.

Though many people continued to work on Serre's problem, little direct progress was made for the next ten years. Then, in 1973, Murthy-Towber, Swan, Roitman, and

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Suslin-Vaserstein all announced important progress. The detailed history is very well portrayed in Bass' paper [2]; the reader may also find there a very complete exposition of the new results. (The material exposed here comes, in fact, from Bass' paper. I am very grateful to Bass for giving me an early copy to study.) Suffice it now to say that we now know that if k is a field, then projective $k[x_1 \ldots x_n]$-modules are free if either

1) $n \leq 3$,

2) $n \leq 4$, char $k \neq 2$,

or 3) $n = 5$, k finite, char $k \neq 2$,

and that the best results of this sort, which apply to much more general rings, are due to Suslin and Vaserstein.

Part of the Suslin-Vaserstein proof is so simple and elegant that it deserves to be seen by everyone. We will give it in full, and sketch what remains to be done for case 2) above.

THEOREM 1 (Roitman-Swan, Suslin): Let $A = k[x_1 \ldots x_n]$.

a) Projective A-modules of rank $\geq n/2 + 1$ are free.

b) Any unimodular row $(f_1, \ldots, f_{r+1})$ of elements of A can be reduced by elementary transformations to $(0, \ldots, 0, 1)$ if $r \geq n/2 + 1$. Here a "unimodular row" is a sequence of elements of A which generate the unit ideal.

Part a) follows from part b) because every projective A-module P is stably free: that is, $P \oplus A \oplus \ldots \oplus A \cong A^m$ for some m, and part b) allows one to "cancel" the copies of A.

For the proof of b) we need one prerequisite:

Stable Range Theorem (Bass [1]): If B is a noetherian ring of dimension d, and $(c_1, \ldots, c_s)$ is a unimodular row of length $s > d + 1$, then there is a unimodular row $(c'_1, \ldots, c'_{s-1})$ with

$$c_i \equiv c'_i \qquad (\mathrm{mod}\ c_s) .$$

Proof of Theorem 1.b): After a change of variables, we may assume that f_{r+1} is monic of degree $p \geq 1$ in x_n. We can now forget all but the last variable and write

$$B = k[x_1 \ldots x_{n-1}] \quad , \quad t = x_n \quad , \quad A = B[t] \quad .$$

Adding multiples of f_{r+1} to the other f_i, we may assume that they have degree $\leq p-1$.

Case 1 $p = 1$: In this case we may write $f_{r+1} = f'_{r+1} + t$, so that $f_1, \ldots, f_r$,

$f'_{r+1} \in B$. The substitution $t = -f'_{r+1}$ shows that $(f_1,\ldots,f_r)$ is a unimodular row, so we can make elementary transformations

$$(f_1,\ldots,f_{r+1}) \longrightarrow (f_1,\ldots,f_r, 1) \longrightarrow (0,\ldots,0,1) \quad .$$

Case 2 $p \geq 2$: We will show how to lower p by 1, thus eventually reducing to case 1. Let $\underline{c} = (c_1,\ldots)$ be the vector of coefficients of degree $\leq p-1$ of the polynomials $f_1,\ldots,f_r$. Write (c) for the ideal generated by the c_i.

If $(c) = B$, then the ideal generated by $f_1,\ldots,f_{r-1}, f_{r+1}$ contains a monic polynomial of degree $p-1$; for, every c_i occurs as the leading coefficient of a polynomial of degree $p-1$. (Hint: use f_{r+1} to kill off terms of degree $\geq p$ in $t^i f_j$.) By elementary transformations we can replace f_r by a polynomial monic of degree $p-1$, and interchange f_{r+1} and f_r, lowering p as required.

If $(c) \neq B$, then, since $f_1,\ldots,f_{r-1}$ are congruent to 0 in $B/_{(c)}[t]$, I assert that the ideal (f_r, f_{r+1}) contains an element $g \equiv 1 \pmod{(c)}$, and $g \in B$.

For, since f_{r+1} is monic, $B[t]/(f_r,f_{r+1})$ is a finitely generated B-module. Also, we have $(c)B[t]/(f_r,f_{r+1}) = B[t]/(f_r,f_{r+1})$, so for some element $h \in (c)$, $(1+h)B[t]/(f_r,f_{r+1}) = 0$. Take $g = 1+h \in (f_r,f_{r+1})$. (One could also use a resultant). Thus the row $(\underline{c},g)$ is unimodular, and it has $p(r-1) + 1 \geq 2(r-1) + 1 \geq 2(\frac{n}{2}) + 1 = n + 1$ elements. By the stable range theorem applied to B, there exists a unimodular row $\underline{c}' = (c'_1 \ldots)$, congruent to $\underline{c} \bmod (g)$. Clearly $\underline{c}'$ will be the row of coefficients of some row of polynomials of degree $\leq p-1$

$$f'_1,\ldots,f'_{r-1}, \text{ with } f'_i \equiv f_i \pmod{f_r,\ f_{r+1}} \ .$$

Thus we can reduce by elementary transformations to the case $(c) = B$. //

Note that we could replace n, in the proof, by the "stable range" of B, and that hypotheses on B much weaker than that it is a polynomial ring over a field will allow us to make a change of variables making f_{r+1} monic.

To finish the proof that projective A-modules are free if $n \leq 4$, we need only deal with modules of rank 2, and it suffices to show that the special linear group $SL_3(A)$ is transitive on $Un_3(A)$, the set of unimodular rows of length 3. Now given $f = (f_1,f_2,f_3)$, unimodular, we may regard it as the matrix of a map

$$A^3 \xrightarrow{f} A \quad .$$

Let P_f be the kernel. From the exactness of the Koszul complex

$$0 \longrightarrow \overset{3}{\Lambda} A^3 \longrightarrow \overset{2}{\Lambda} A^3 \xrightarrow{\partial} A^3 \xrightarrow{f} A \quad ,$$

and the easy computation $\partial(\overset{2}{\Lambda} P_f) = 0$ we get a map

$$\overset{2}{\Lambda} P_f \longrightarrow \overset{3}{\Lambda} A^3 \cong A \text{ .} \qquad \text{(This is in fact an isomorphism).}$$

Such a map may be regarded as an alternating form on P. We thus get a map $Un_3(A) \longrightarrow KSp_0(A)$, the Grothendieck group of projective modules with alternating forms. It turns out that the orbits of $SL_3(A)$ are identified under this map, so we get a map

$$Un_3(A)/_{SL_3A} \xrightarrow{\varphi} KSp_0A \text{ .}$$

Recall that for any r, $E_r(A)$ is the subgroup of $SL_r(A)$ generated by the elementary transformations. Theorem 1b asserts that $E_{r+1}(A)$ is transitive on $Un_{r+1}(A)$ for all $r \geq n/2 + 1$. Thus we can apply the following theorem of Vaserstein to $A = k[x_1,\dots,x_n]$ in the case $n \leq 4$:

<u>Theorem</u> (Vaserstein): For any commutative ring A, if $E_{r+1}(A)$ is transitive on $Un_{r+1}(A)$ for all $r \geq 3$, then φ is injective.

Note that P_f is not only projective, it is stably free; thus the image of φ is in the kernel $W(A)$ of the composite map

$$\begin{array}{ccc} KSp_0(A) & \xrightarrow{\text{forgetful map}} & K_0(A) \\ & & \downarrow \\ & & K_0(A)/\{\text{stably free modules}\} \\ & & \parallel \\ & & \tilde{K}_0(A) \text{ .} \end{array}$$

Now we want to compute $W(A)$:

<u>Theorem</u> (Karoubi [3]): If $A = B[t]$, and $\frac{1}{2} \in B$, then $W(A) = W(B)$.

Thus, in our case, $A = k[x_1,\dots,x_4]$, $SL_3(A)$ is transitive on $Un_3(A)$, so $W(A) = 0$ and projectives of rank 2 are free. Thus all projectives over $k[x_1,\dots,x_4]$ are free.

REFERENCES

1) Bass, H.; K-theory and stable algebra. Publ. I.H.E.S. 22 (1964), 5-60.

2) Bass, H.; Liberation des modules projectifs sur certains anneaux de polynômes. Sem. Bourb. 1974.

3) Karoubi, M.; Periodicité de la K-theorie hermitienne, Proc. Batelle Conf. on Algebraic K-theory, vol. III, Springer Lecture Notes 343, Springer-Varlag, N.Y. 1973.

4) Serre, J.-P.; Faisceaux algebriques coherents. Ann. Math. 61 (1955), 197-278.

II. SOME RECENT PROGRESS ON THE "HOMOLOGICAL CONJECTURES" IN COMMUTATIVE RING THEORY

I. Survey of the situation

Probably the biggest advance in commutative rings in the 1950's was the introduction of homological methods, mostly by Auslander, Buchsbaum, and Serre, leading to the solution to a number of old problems (regular local rings are factorial, etc.). A few of the problems as to the nature of the new homological machines have never been completely resolved, though there has been quite a bit of work and progress. Here are two "central" examples:

a) The Multiplicity Problem: In his 1957 notes "Algebre Locale-Multiplicités", which consolidated and explained the new homological process, Serre gave a homological treatment of intersection multiplicities. He proved, for regular local rings R containing a field (and more generally for "unramified" regular local rings), that if M and N are two modules such that $M \otimes N$ has finite length, then, writing $\dim M$ for $\dim \operatorname{supp} M$, and $\ell(T)$ for the length of an R-module of finite length,

i) $\dim M + \dim N \leq \dim R$

ii) $\chi(M,N) = \sum_{i\geq 0} (-1)^i \ell(\operatorname{Tor}_i(M,N)) \geq 0$

iii) $\chi(M,N) = 0$ if and only if $\dim M + \dim N < \dim R$.

(The central case, and that corresponding to classical intersection theory, is that in which $M = R/P$, $N = R/Q$, and P and Q are prime ideals).

The problem that remains is to extend this theorem to all regular local rings, or, most optimistically, to prove that it holds for any local ring R, provided only that p.d. M , the projective dimension of M, is finite.

b) The Rigidity of Tor: Auslander conjectured that, if R is a regular local ring, and if M and N are finitely generated R-modules, such that $\operatorname{Tor}_1(M,N) = 0$, then $\operatorname{Tor}_i(M,N) = 0$ for every $i > 0$; this was subsequently proved by Lichtenbaum. Again, the remaining question is: is this rigidity property true when R is any local ring, assuming p.d. $M < \infty$?

There has been some progress on problem a) since 1957, for example by Malliavin and Hochster and others, and an important step toward the "optimistic" version has recently been taken by Peskine-Szpiro [3]. The biggest progress, however, has been made on our understanding of the consequences and relations of problem b). It is this work that I want to tell you about, especially some new work of Hochster on a conjecture which has among its consequences most of the well-known consequences of b).

The "modern" development of the problem seems to me to begin with the Thesis of

Peskine-Szpiro [2], in which they exhibited a consequence of b), and verified it in a large number of cases. It is as follows:

c) The Intersection property for a local ring R: If M and N are finitely generated R-modules and $M \otimes N$ has finite length, then

$$\text{p.d. } M \geq \dim N \text{ .}$$

Conjecture c) is actually a consequence of conjecture ai) in the regular, or even Cohen-Macaulay case. To see this, note that c) is vacuous if $\text{p.d. } M = \infty$, while if $\text{p.d. } M < \infty$, then $\text{p.d. } M = \text{depth } R - \text{depth } M$, so c) becomes

$$\text{depth } R \geq \dim N + \text{depth } M \text{ .}$$

(Here, depth M is the maximal length of an M-sequence. Note that $\text{depth } M \leq \dim M$ always holds -- in fact depth M is bounded by the dimension of any associated prime of M -- and that depth $R = \dim R$ if R is Cohen-Macaulay.)

One reason for the significance of the intersection property becomes clear if, with Hochster, we note that it is equivalent to an extension of the Krull principal ideal theorem:

d) Homological height conjecture: Let R be a local ring, S an R-algebra, and M a finitely generated R-module. Then $\text{p.d. } M \geq \text{height } (\text{annihilator}_S(S \otimes M))$.

(To get the usual version of the principal ideal theorem, take R to be a localization of $Z[t]$ at a prime containing t, and take $M = R/(t)$.)

Another reason why the intersection property is important is that though it seems to be much easier to check than the rigidity conjecture, its truth would imply most of the known consequences of the rigidity conjecture, such as the following conjectures:

i) (Auslander) If M is a finitely generated module over the local ring R, with

$$\text{p.d. } M < \infty \text{ ,}$$

and if $x \in R$ is a nonzerodivisor on M, then x is a nonzerodivisor on R.

ii) (Bass) If R is a local ring which possesses a finitely generated module of finite injective dimension, then R is Cohen-Macaulay.

Within the last year, a new step in the direction of these conjectures has been taken by Hochster, and it is his work that I wish to describe here. A further definition is necessary: If M is a (possibly not finitely generated) R-module, then a sequence of elements $(x_1,\ldots,x_n)$ of R is an M-sequence if

i) $(x_1,\ldots,x_n)M \neq M$

ii) x_i is a nonzerodivisor on $M/(x_1,\ldots,x_{i-1})M$ for $i = 1,\ldots,n$.

Recall that if R is a local ring of dimension n with maximal ideal J, then a system of parameters is a sequence of n elements $x_1,\ldots,x_n \in R$ such that the ideal $(x_1,\ldots,x_n)$ is J-primary.

<u>Theorem</u> (Hochster): If R is a local ring containing a field, and if $x_1,\ldots,x_n$ is a system of parameters in R, then there exists an R-module M such that $x_1,\ldots,x_n$ is an M-sequence.

(More generally, this works even if only R_{red} contains a field.) The interest for us of this theorem is clear from the following:

<u>Proposition</u> (Hochster): If R is a local ring, and if for each prime ideal P of R, every system of parameters in R/P is an h-sequence for some R/P-module h then the intersection property holds for R.

<u>Proof</u>: Assume that M and N are finitely generated modules such that $M \otimes N$ has finite length, and let $P \in \operatorname{supp} N$ be such that $\dim R/P = \dim N$; it is easy to see that $M \otimes R/P$ has finite length. Since $P + \operatorname{ann} M$ is primary to the maximal ideal of R, we may choose a system of parameters $(x_1,\ldots,x_d)$ for R/P contained in $\operatorname{ann} M$, and let h be an R/P module such that $(x_1,\ldots,x_d)$ is an h-sequence.

Note that

$$\begin{aligned}\operatorname{Tor}_0^R(M,h) &= M \otimes_R h \\ &= M/PM \otimes_{R/P+(x_1,\ldots,x_d)} h/(x_1,\ldots,x_d)h \\ &\neq 0\ ,\end{aligned}$$

since both tensor factors are $\neq 0$ and the maximal ideal of $R/P + (x_1,\ldots,x_d)$ is nilpotent. Thus we may choose $i \geq 0$ maximal with respect to the condition

$$\operatorname{Tor}_i^R(M,h) \neq 0\ .$$

Since $\operatorname{Tor}_*^R(M,-)$ is annihilated by each x_i, the long exact sequences associated to the sequences

$$0 \longrightarrow h(x_1,\ldots,x_i)h \xrightarrow{x_{i+1}} h(x_1,\ldots,x_i)h \longrightarrow h/(x_1,\ldots,x_{i+1})h \longrightarrow 0$$

show that

$$\operatorname{Tor}_{i+d}^R(M,h/(x_1,\ldots,x_d)h \cong \operatorname{Tor}_i(M,h) \neq 0\ .$$

Thus $\text{p.d. } M \geq i + d \geq d = \dim N$. //

The existence of Macaulay modules can be used for other things besides the intersection property. One nice example comes from the following:

Theorem (Hochster): Let R be a regular local ring, and let $S \supseteq R$ be a finite R-algebra. Suppose that there is an S-module M such that some regular system of parameters for R is an M sequence. Then R is a direct summand of S (as R-modules).

Corollary: If R is a regular local ring containing a field, and $S \supseteq R$ is a finite algebra, then R is a direct summand of S.

(In fact Hochster has given a proof of this corollary which is independent of the existence of Macaulay modules.)

II. Sketch of the construction of Cohen-Macaulay modules

Hochster's construction of Cohen-Macaulay modules is childishly simple. Furthermore, it is easy to show that if Cohen-Macaulay modules exist at all, then the construction does lead to them. Nevertheless, Hochster's proof that the construction does work is quite intricate, and I'll confine myself to describing the more canonical looking (and therefore easier) portion.

A.) The construction: Fix a local ring R and a system of parameters $x_1,\dots,x_n \in R$. If M is any R-module, and if $m \in M$ is such that

$$x_{k+1}m \in (x_1,\dots,x_k)M \quad \text{for some } k < n ,$$

we may define a new module M', called a modification of M, by

$$M' = \frac{M \oplus R^k}{R(m,-x_1,-x_2,\dots,-x_k)} .$$

The natural map $M \longrightarrow M'$ is clearly universal for maps $M \xrightarrow{\varphi} N$ such that $\varphi(m) \in (x_1,\dots,x_k)N$. Now start with $M = R$, and consider, for each k, a finite set of elements in R which generates

$$((x_1,\dots,x_k)M \ : \ x_{k+1})/(x_1,\dots,x_k)M .$$

Taking finitely many successive modifications, we may construct a module M_1 and a map $\varphi_1 : R \longrightarrow M_1$ so that

$$\varphi((x_1,\dots,x_k)M:x_{k+1}) = (x_1,\dots,x_k)M_1 .$$

Repeating this procedure countably many times, and taking the limit, we will obtain a module M_∞ with the property that for each $k < n$,

$$(x_1,\dots,x_k)M_\infty : x_{k+1} = (x_1,\dots,x_k)M_\infty .$$

The elements $x_1,\dots,x_n$ will be an M_∞-sequence if and only if we have in addition

$$(x_1, \ldots, x_n)M_\infty \neq M_\infty .$$

We can actually improve this assertion considerably:

<u>Proposition</u>: With the notation above, let $\varphi_\infty : R \longrightarrow M$ be the natural map. There exists an R-module N such that $x_1, \ldots, x_n$ is an N-sequence if and only if $\varphi_\infty(1) \notin (x_1, \ldots, x_n)M_\infty$ (in which case $x_1, \ldots, x_n$ is an M_∞- sequence).

<u>Proof</u>: We need only show that if $(x_1, \ldots, x_n)$ is an N-sequence, then $\varphi_\infty(1) \notin (x_1, \ldots, x_n)M_\infty$. Let $e \in N - (x_1, \ldots, x_n)N$, and let $f: R \longrightarrow N$ be defined by $f(1) = e$. By the universal property of modifications, there exists an extension $\psi: M_\infty \longrightarrow N$.

$$\begin{tikzcd} R \arrow[r, "\varphi_\infty"] \arrow[d, "f"'] & M_\infty \arrow[dl, dashed, "\psi"] \\ N & \end{tikzcd} .$$

Thus if $\varphi_\infty(1) \in (x_1, \ldots, x_n)M_\infty$, we would have

$$\psi\varphi_\infty(1) = f(1) = e \in (x_1, \ldots, x_n)N ,$$

a contradiction. //

B.) The proof: We now see that to prove the existence of a module M on which a given system of parameters is an M-sequence, it suffices (and is also necessary) to show:

$$(*) \left\{ \begin{array}{l} \text{No sequence of modifications} \\ \qquad \begin{tikzcd} R \arrow[r] \arrow[rrr, bend right, "\varphi"'] & M_1 \arrow[r] & \cdots \arrow[r] & M_i \end{tikzcd} \\ \text{has the property that } \varphi(1) \in (x_1, \ldots, x_n)M_i . \end{array} \right.$$

Hochster's proof that this is so for a local ring R containing a field k has two parts: a proof when $\operatorname{char} k = p > 0$, which uses the Frobenius homomorphism, and a reduction to this case. Hochster's paper (see the references) contains a very illuminating and fairly full account of these processes, the second of which involves Artin's Approximation Theorem. Here we will content ourselves with giving the "punch line" -- the case in which R is a complete local domain of characteristic $p > 0$.

The usual proof that any variety maps birationally to a hypersurface can be applied to show that, at least after extending R slightly, we may assume $R \supseteq R_0 \ni x_1,\dots,x_n$, where R_0 is a Cohen-Macaulay local ring, $x_1,\dots,x_n$ are an R_0-sequence, and R is a finitely generated R_0-submodule of the quotient field of R_0. Thus there exists an element $c \in R_0$ such that $cR \subseteq R_0$.

We claim that for any integer $t \geq 1$, and any $k < n$,

$$c[(x_1^t,\dots,x_k^t) : x_{k+1}^t] \subseteq (x_1^t,\dots,x_k^t) .$$

(Here all ideals are to be thought of as ideals of R.) For, if $ax_{k+1}^t \in (x_1^t,\dots,x_k^t)$, then $cax_{k+1}^t \in \Sigma_1^k R_0 x_i^t$, so $ca \in \Sigma R_0 x_i^t$, since $x_1^t,\dots,x_{k+1}^t$ is an R_0-sequence.

Now suppose that

$$*) \qquad R \xrightarrow{\varphi_1} M_1 \xrightarrow{\varphi_2} M_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_n} M_n$$

is a sequence of modifications with respect to the system of parameters $x_1^t,\dots,x_n^t$, for any t. We will show that there are maps $\psi_i : M_i \longrightarrow R$ making the following diagram commute:

$$(**) \qquad \begin{array}{ccccccccc} R & \xrightarrow{\varphi_1} & M_1 & \xrightarrow{\varphi_2} & M_2 & \longrightarrow & \cdots & \xrightarrow{\varphi_n} & M_n \\ \| & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & & & \downarrow{\scriptstyle \psi_n} \\ R & \xrightarrow[c]{} & R & \xrightarrow[c]{} & R & \xrightarrow[c]{} & \cdots & \xrightarrow[c]{} & R \end{array}$$

In fact, given $M \xrightarrow{\psi} R$ and an element $m \in M$ such that $x_{k+1}^t m \in (x_1^t,\dots,x_k^t)$, we have $x_{k+1}^t \psi(m) \in (x_1^t,\dots,x_k^t)$, and thus $c\psi(m) \in (x_1^t,\dots,x_k^t)$. If $\varphi : M \longrightarrow M'$ is a modification corresponding to the equation $x_{k+1}^t m \in (x_1^t,\dots,x_k^t)$, then by the universal property of φ, there exists a map ψ' making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi'} \\ R & \xrightarrow[c]{} & R \end{array}$$

commute.

Suppose now that $t = 1$, and $\varphi_n \cdots \varphi_2\varphi_1(1) \in (x_1,\dots,x_n)$. Raising all the elements occuring in the construction of $*)$ to the p^m power, we see that we get a sequence of modifications

$$*_m) \qquad R \xrightarrow{\varphi_1^{(m)}} M_1^{(m)} \longrightarrow \cdots \xrightarrow{\varphi_n^{(m)}} M_n^{(m)} ,$$

with respect to the system of parameters $x_1^{p^m}, \ldots, x_n^{p^m}$. These all have the property that

$$\varphi_n^{(m)} \cdot \ldots \cdot \varphi_1^{(m)} \in (x_1^{p^m}, \ldots, x_n^{p^m}) .$$

However, by **) we see that then

$$c^n \cdot 1 \in (x_1^{p^m}, \ldots, x_n^{p^m}) .$$

Since this works for every m, we get a contradiction. Thus the modification procedure really does lead to a module M such that $x_1, \ldots, x_n$ is an M-sequence.

References

There are two excellent sources for a more detailed treatment of this material (and for a more complete bibliography!). They are

1) Hochster, M.: "Topics in the Homological Theory of Modules over Commutative Rings" (A.M.S. Regional Conference, June 1974, Lincoln, Nebraska).

2) Peskine, C., and Szpiro, L.: Dimension Projective Finie et Cohomologie Locale. Publ. Math. I.H.E.S., 42 (1973) Paris.

Too recent to be mentioned in the above is the following very interesting note on the intersection conjectures:

3) ________, ____________: Syzygies et Multiplicités. C. R. Acad. Sci., Paris, 27 May 1974.

III. "PERFECT IDEALS OF CODIMENSION 2 ARE DETERMINANTAL"- SOME APPLICATIONS

One of the active areas in commutative algebra is concerned with the structure of finite free resolutions and the consequences of that structure for ideal and module theory. Rather than go into the recent work on free resolutions here, I want to exhibit the archetypal result in this field (Theorem 1, below), and two of its more unusual applications (The reader who wishes a survey of work on free resolutions may find [5] useful.). The first of these applications, a theorem on factoriality, is not so new, but is a kind of mathematical "chestnut." The second represents our best current information on the Zariski-Lipman conjecture.

I): The structure of perfect ideals of codimension 2.

Hilbert seems to have arrived at the idea of classifying ideals in polynomial rings by their projective dimensions as soon as he had proved that their dimensions were finite [5]. He gives (as an example of the use of the Hilbert function) a special case of the following theorem, which gives a generic form for ideals of projective dimension 1 and their resolutions. For convenience we will work not with an ideal I in a ring R, but with the cyclic module R/I. We will always assume that R is a noetherian local ring, and that modules are finitely generated.

Theorem 1: Let

$$\underline{F} : 0 \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

be an exact sequence of R-modules, with F_1 and F_2 free, and rank $F_1 = n$. Then rank $F_2 = n-1$, and φ_1 can be factorized as

$$\begin{tikzcd} F_1 \arrow[rr, "\varphi_1"] \arrow[d, equal, "\wr"] & & R \\ \overset{n-1}{\Lambda} F_1^* \arrow[rr, "\overset{n-1}{\Lambda} \varphi_2^*"'] & & \overset{n-1}{\Lambda} F_2^* \cong R \arrow[u, "a"'] \end{tikzcd} ,$$

where $a \in R$ is a nonzerodivisor.

If we assume that I has height at least 2 (and thus exactly 2, since p.d. $R/I \geq$ ht I always holds), it follows by Krull's principal ideal theorem that a is a unit, which we may ignore. If we choose a basis for F_1, the map $\overset{n-1}{\Lambda} \varphi_2^*$ will have as its n coordinates the $n-1 \times n-1$ subdeterminants of φ_2. Remembering that an ideal I is said to be perfect if ht I = p.d. R/I, we can restate Theorem 1 more colloquially as "Perfect ideals of height 2 are determinantal."

Theorem 1 has a very useful converse:

Theorem 2: Let $\varphi: R^{n-1} \longrightarrow R^n$ be a map, and let

$$I = \text{image } (\overset{n-1}{\Lambda} \varphi^*: \overset{n-1}{\Lambda} R^{n*} \longrightarrow \overset{n-1}{\Lambda} R^{n-1*} \cong R) ,$$

the "ideal of $n-1 \times n-1$ subdeterminants of φ." Suppose I is contained in the maximal ideal of R. If $\text{depth}_I R \geq 2$, then I is perfect, and $\text{depth}_I R = 2$. Furthermore, if ψ is defined to be the composite

$$\begin{tikzcd} R^n \cong \overset{n-1}{\Lambda} R^{n*} \arrow[r, "\overset{n-1}{\Lambda} \varphi^*"] \arrow[rr, bend right, "\psi"'] & \overset{n-1}{\Lambda} R^{n-1*} \cong R & \end{tikzcd} ,$$

then the sequence

$$*) \quad 0 \longrightarrow R^{n-1} \xrightarrow{\varphi} R^n \xrightarrow{\psi} R \longrightarrow R/I \longrightarrow 0$$

is exact. If the elements of φ are in the maximal ideal of R, then $*)$ is the minimal free resolution of R/I.

Here we are using the familiar isomorphisms $\overset{k}{\Lambda} R^n \cong \overset{n-k}{\Lambda} R^{n*}$, which are defined by the choice of a basis for R^n though, up to multiplication by a unit, it is independent of this choice. The fact that $*)$ is a complex comes from the fact that the i^{th} component of the map $\psi\varphi$ is the $n \times n$ determinant formed from the $n \times n-1$ matrix of φ by repeating the i^{th} column. It should be remarked that our various hypotheses— the local and noetherian properties of R and the freeness of the F_i—could be eliminated; the theorem really concerns arbitrary finitely presented ideals of projective dimension 1.

Theorems 1 and 2 were proved in essentially the form given here by L. Burch [4]; the shortest proof is surely that of Kaplansky [8]. For various generalizations of the theorem to longer free resolutions, see [3]. Despite the existence of Kaplansky's bare-hands proof, I would like to sketch a slightly less elementary proof which, however, generalizes better. The point is that any good criterion for the exactness of a finite complex will yield a proof of Theorems 1 and 2. For, to prove Theorem 2, it suffices to show that the complex

$$0 \longrightarrow R^{n-1} \xrightarrow{\varphi} R^n \xrightarrow{\psi} R$$

considered there is exact. To prove Theorem 1, on the other hand, it would be enough to show that the dual complex

$$0 \longrightarrow R^* \xrightarrow{\psi^*} R^{n*} \xrightarrow{\varphi_2^*} R^{n-1*} ,$$

(where ψ is constructed from $\varphi_2 = \varphi$ as in Theorem 2) is exact; for then the comparison theorem for resolutions would yield a map $a: R^* \longrightarrow R^*$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & R^* \arrow[r, "\varphi_1^*"] \arrow[d, "a"'] & R^{n*} \arrow[r, "\varphi_2^*"] \arrow[d, equal] & R^{n-1*} \arrow[d, equal] \\
0 \arrow[r] & R^* \arrow[r, "\psi^*"'] & R^{n*} \arrow[r, "\varphi_2^*"'] & R^{n-1*}
\end{tikzcd}$$

commute. (It is then easy to see that a is induced by multiplication by a non-zero-divisor).

To state a suitable exactness criterion, we need two preliminaries: Let $\varphi: F \longrightarrow G$ be a map of free modules. Define rank $\varphi = r$ to be the largest integer i such that $\overset{i}{\Lambda} \varphi \neq 0$, and let $I(\varphi)$ be the ideal generated by the $r \times r$ minors of a matrix representing φ. (Invariantly put, $I(\varphi)$ is the image of the natural map induced by $\overset{r}{\wedge} G^* \otimes \overset{r}{\wedge} F \longrightarrow R$ induced by φ.) We can now state an exactness criterion which is ample for the proofs of Theorems 1 and 2; it is a slightly special case of the main result of [2].

Theorem: Let $F\colon 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$ be a complex of finitely generated free modules over a noetherian ring R. The complex F is exact if and only if

1) $\operatorname{rank} \varphi_r + \operatorname{rank} \varphi_{r+1} = \operatorname{rank} F_k \qquad (k = 1,\ldots,n)$

and 2) for each k, $\operatorname{depth}_{I(\varphi_k)} R \geq k$.

(Note that by convention, $\operatorname{depth}_R R = \infty$.)

II): Two Applications.

Several people at this conference have already spoken about work on "subvarieties of low codimension," and the two theorems just described are important for some of this work. Another area of usefulness of the Hilbert-Burch theorems which has not yet been mentioned is deformation theory. But in this talk I want to speak of two other, somewhat more novel applications.

A) The local factoriality of surfaces in 3-space (after Andreotti-Salmon [1]).

We will say that an element f of a local ring R is a determinant if f is the determinant of some $n \times n$ matrix, $n \geq 2$, with entries in the maximal ideal of R. Equivalently, f could be the determinant of an $n \times n$ matrix whose $n-1 \times n-1$ minors do not generate the unit ideal of R.

Theorem: Let R be a 3-dimensional regular local ring, and let $f \in R$ be a prime element. Then $R/(f)$ is factorial if and only if f is not a determinant.

Proof: The factoriality of a 2-dimensional local domain S is equivalent to the statement that every unmixed ideal of dimension 1 of S is principal. Suppose that f is a determinant -- say $f = \det A$, where A is an $n \times n$ matrix whose entries are in the maximal ideal, $n \geq 2$. Let B be the matrix obtained from A by omitting the first column. By the Laplace expansion of $\det A$ along the first column, f is an element of the ideal I of $n-1 \times n-1$ minors of B. Moreover, since the elements of that first column are in the maximal ideal J of R,

$$f \in JI\ ;$$

in particular $I \neq (f)$, and since (f) is a prime, we see that $\operatorname{depth}_I R \geq 2$.

Theorem 2 now shows that the free resolution of R/I over R is

$$0 \longrightarrow R^{n-1} \xrightarrow{B} R^n \xrightarrow{\psi} R \longrightarrow R/I \longrightarrow 0\ ,$$

where ψ is obtained from $B = \varphi$ as in Theorem 2. Since the entries of B are in the maximal ideal, I is minimally generated by $n \geq 2$ elements, and since $f \in JI$, $I/(f)$ is also minimally generated by n elements. Since I is perfect of codimension 2 in R, it is unmixed, and $I/(f)$ is of dimension 1 in $R/(f)$. By the remark at the beginning, the 2-dimensional domain $R/(f)$ is not factorial.

Conversely, suppose that f is not a determinant in R; we will show that if $I/(f)$ is an unmixed, one-dimensional ideal in $R/(f)$, then I can be generated by one element in addition to f. Let $f, g_1, \ldots, g_k$ be a set of generators for I. Since I is unmixed of dimension 1 in R, it is perfect of codimension 2; applying Theorem 1, and suppressing a (which must be a unit in this case) we see that I has a resolution

$$0 \longrightarrow R^k \xrightarrow{\varphi_2} R^{k+1} \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0 \quad ,$$

and that the determinant of some $k \times k$ submatrix A of φ_2 is f. Since f "is not a determinant," some $k-1 \times k-1$ minor of φ is a unit. Thus after a change of basis in R^k and R^{k+1}, the matrix of φ_2 has the form

$$\begin{array}{cc} & \overbrace{\qquad\qquad\qquad}^{k-1} \\ k-1\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \left(\begin{array}{cccccc|cc} 1 & & & & & & 0 & 0 \\ & 1 & & 0 & & & \cdot & \cdot \\ & & \ddots & & & & \cdot & \cdot \\ & & & 1 & & & \cdot & \cdot \\ & 0 & & & 1 & & & \\ & & & & & 1 & 0 & 0 \\ \hline 0 & \ldots & & & & 0 & f & g \end{array}\right) \end{array} \quad ,$$

and $I = (f,g)$ is generated by one element, g, modulo f. //

B) The Zariski-Lipman conjecture for hypersurfaces (after Scheja-Storch [11,12]).

The Zariski-Lipman conjecture is the following:

Let R be a local domain essentially of finite type over a field k of characteristic 0. If the module $\mathrm{Hom}_R(\Omega_{R/k}, R)$ of k-derivations from R to R, is a free module, then R is regular. (Of course R is regular if $\Omega_{R/k}$ is free). Lipman proved in his Thesis [9] that under the hypothesis of the conjecture, R is at least normal. Since that time, Moen [10] gave a proof of the conjecture in case R is both the coordinate ring of a cone and a complete intersection, and his proof was subsequently simplified by Hochster [7]. (This setup is rather special, since a smooth cone is just a linear space!) Most recently, Scheja and Storch have proved the conjecture under the hypothesis that R is the coordinate ring of a hypersurface (and under slightly less restrictive hypotheses than those in the conjecture as given above). We will give a sketch of their proof.

We will assume for simplicity that the residue class field of R is k, and we will write

$$R = S/(f) \ , \ S = k[x_1,\ldots,x_n]_J \ ,$$

where J is the maximal ideal $(x_1,\ldots,x_n)$. The map of k-algebras $S \longrightarrow R$ gives rise to an exact sequence

$$(f)/(f^2) \longrightarrow \Omega_{S/k} \otimes R \longrightarrow \Omega_{R/k} \longrightarrow 0 \ .$$

Since $f/(f^2)$ is a free R-module of rank 1, and $\Omega_{S/k}$ is free of rank n on the elements dx_i, we may rewrite this as

$$*) \qquad R \xrightarrow[\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial x_n} \end{pmatrix}]{} R^n \longrightarrow \Omega_{R/k} \longrightarrow 0 \ .$$

By hypothesis, $\Omega^*_{R/k} = \mathrm{Hom}(\Omega_{R/k},R)$ is free, and since R is "generically smooth of dimension n-1", we must have $\Omega^*_{R/k} \cong R^{n-1*}$. Thus, dualizing $*)$, we obtain an exact sequence

$$0 \longrightarrow R^{n-1*} \xrightarrow[\varphi]{} R^{n*} \xrightarrow[(\frac{\partial f}{\partial x})]{} R \ .$$

We may now apply Theorem 1 to this exact sequence. By Lipman's theorem, R is normal, and thus nonsingular in codimension 1 (it is easy to prove this directly), so the depth of the ideal generated by $(\frac{\partial f}{\partial x_i})$ in R is at least 2. Thus the element a of Theorem 1 must be a unit, and we ignore it. The conclusion of Theorem 1 is now that $\frac{\partial f}{\partial x_i}$ is (up to sign) the determinant obtained from φ be removing the i^{th} row.

We now choose any matrix $\tilde{\varphi}$ with entries in S which reduces to φ modulo f, and let Δ_i be the determinant obtained from $\tilde{\varphi}$ by omitting the i^{th} row. Since $\frac{\partial f}{\partial x_i} = \pm \Delta_i$ (mod f), we have

$$\#) \qquad df \in (\Delta_1,\ldots,\Delta_n, f)\, \Omega_{S/k} \ .$$

We assert that #) implies that $f \in \sqrt{(\Delta_1,\ldots,\Delta_n)}$. The condition # is stable under "base changes" $S \xrightarrow{\alpha} S'$, and if $f \notin \sqrt{(\Delta_1,\ldots,\Delta_n)}$, we could choose S' to be a formal power series ring in one variable z, and α a map such that

$$\alpha((\Delta_1,\ldots,\Delta_n)) = 0 \qquad \alpha(f) \neq 0 \ .$$

However, we would then get

$$df \in f\,\Omega_{S'/k} \qquad (\Omega_{S'/k} \text{ is the module of } \underline{\text{continuous}} \text{ differentials!})$$

which is a contradiction because the order of the power series $\frac{df}{\partial z}$ is lower than the order of f. Thus $f \in \sqrt{(\Delta_1,\ldots,\Delta_n)}$.

If now $R = S/f$ is not regular, the ideal generated by the elements $\frac{\partial f}{\partial x_i}$ must be in the maximal ideal. Thus the $k-1 \times k-1$ subdeterminants of φ_1 and thus of $\tilde{\varphi}$, are in the maximal ideal. By Theorem 2, the height of the ideal $(\Delta_1,\ldots,\Delta_n)$ is ≤ 2. Since $f \in \sqrt{(\Delta_1,\ldots,\Delta_n)}$, the height in R of $\sqrt{(\Delta_1,\ldots,\Delta_n)}/(f) = \sqrt{\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)}/f$ is ≤ 1, contradicting the fact that, since R is normal, it is nonsingular in codimension 2. Thus R must be regular. //

REFERENCES

1) Andreotti, A., and Salmon, P.; Annelli con unica decomponibilita in fattori primi ed un problema di intersezioni complete. Monatsh. für Math. 61 (1957), 97-142.

2) Buchsbaum, D.A., and Eisenbud, D.; What makes a complex exact?, J. Alg. 25 (1973), 259-268.

3) ________________; Some structure theorems for finite free resolutions, Adv. in Math. 12 (1974), 84-139.

4) Burch, L.; On ideals of finite homological dimension in local rings, Proc. Cam. Phil. Soc. 64 (1968), 949-952.

5) Eisenbud, D.; A survey of some results on finite free resolutions. Proceedings of the International Congress of Mathematicians, 1974.

6) Hilbert, D.; Über die Theorie der algebraischen formen, Math. Ann. 36 (1890), 473-534.

7) Hochster, M.;

8) Kaplansky, I.; "Commutative Rings," Allyn and Bacon, Boston, 1970.

9) Lipman, J.; Face derivation modules on algebraic varieties, Am. J. Math. 87 (1965), 874-898.

10) Moen,

11) Scheja, G., and Storch, U.; Über differentielle Abhängigkeit bei idealen analytischer algebren, Math. Z. 114 (1970), 101-112.

12) ________________________; Differentielle Eigenschaften der Lokaliserungen analytischer Algebren. Math. Ann. 197 (1972), 137-170.