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Generic Free Resolutions and a Family of Generically Perfect Ideals

DAVID A. BUCHSBAUM AND DAVID EISENBUD*

Department of Mathematics, Brandeis University, Waltham, Mass. 02154

INTRODUCTION

Beginning with Hilbert's construction of what is now called the Koszul complex [18], the study of finite free resolutions of modules over commutative rings has always proceeded by a study of certain particular generic resolutions. This has led to information about the structure of all finite free resolutions, as in [5] and [11], and to theorems on the structure and deformation of certain classes of "generically perfect" ideals and other ideals whose resolutions are of a known type [5, 6, 9, 15, 22, 24, 26].

In this paper we will describe some new classes of finite free resolutions and generically perfect ideals. Under "generic" circumstances we will construct the minimal free resolution of the cokernel of a map of the form $\wedge^k \phi$ or $S_k\phi$, where $\phi: F \to G$ is a map between free modules F and G over a noetherian commutative ring, with $\operatorname{rank} F \geqslant \operatorname{rank} G$, and where $\wedge^k$ and S_k denote the kth exterior and symmetric powers, respectively. We will also describe a family of finite complexes associated with the $(n-1)$st order minors of an $n \times n$ matrix (a *minor* is the determinant of a submatrix). Finally, we will consider a class of ideals that is related to inclusions of one ideal generated by an R-sequence in another. This class includes, for example, the ideal defining the singular locus of a projective algebraic variety that is a complete intersection in $\mathbf{P}^m$.

Our main innovation is the construction and use of a (doubly indexed) family of "multilinear" functors $L_p{}^q$ defined on finitely generated free modules, which includes both the symmetric and exterior powers. For $q \geqslant 1$, $L_1{}^q \cong \wedge^q$, while for $p \geqslant 0$, $L_p{}^1 \cong S_p$. These arise naturally in the resolutions of cokernels of the maps $\wedge^k \phi$ and $S_k\phi$; and it turns out more generally that the free modules occuring in the generic minimal free resolutions of the cokernel of a map of the form $L_p{}^q\phi$ can all be expressed in terms of tensor products of the form $L_p{}^qF \otimes (L_r{}^sG)^*$.

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To describe our other results in more detail, we fix some notation: Let R be a noetherian ring, and let $\phi: F \to G$ be a map of finitely generated free R-modules (as above) with rank $F = m \geqslant$ rank $G = n$.

The ideals whose generic perfection we will establish are defined as follows. Let $a \in F$ be an element, and write $b = \phi(a)$. We will think of F and G being equipped with bases, so that we can speak of the ideal $I_n(\phi)$ generated by the $n \times n$ minors of ϕ, and the ideal $I(b)$ generated by the coordinates of b. (A basis-free treatment is given in Section 5.) Let $I = I_n(\phi) + I(b)$. Recall that the *grade* of an ideal $J \subseteq R$ is the length of a maximal R-sequence in J and that J is *perfect* if $pd\, R/J =$ grade J, where $pd\, R/J$ is the projective dimension of R/J as an R-module. The ideal I defined above is then perfect whenever its grade is m, the maximum possible. This we prove by constructing a complex $\mathbf{K}(\phi, a)$ of length m, which is a resolution of R/I whenever grade $I = m$. This complex appears as the total complex of a double complex whose rows are the resolution of coker $\wedge^q \phi$ for $1 \leqslant q \leqslant n$. We are grateful to Herzog, who inspired our work on these ideals by showing us a preprint of his [16] in which he treats the ideal I as above in the cases $m = n$ and $m = n + 1$.

To see the relationship of these ideals to projective complete intersections, let $V \subseteq \mathbf{P}_k^{m-1}$ be a complete intersection of codimension n, and assume char $k = 0$. If a is the element of the free module $F = (k[X_0, ..., X_{m-1}])^m$ whose coordinates are $X_0, ..., X_{m-1}$, and if ϕ is the jacobian matrix of V, then the coordinates of $b = \phi(a)$ generate the homogeneous ideal of V, and the ideal $I = I_n(\phi) + I(b)$ defines the singular locus of V. In this situation $\mathbf{K}(\phi, a)$ is exact if and only if V is nonsingular and its homology should be an interesting measure of the singularity of V.

Before describing our results on complexes associated with the $(n - 1)$st order minors of an $n \times n$ matrix, we digress to sketch what is already known.

With a few exceptions [5, 6, 16, 27], the known classes of finite free resolutions are easily catalogued. Let R and $\phi: F \to G$ be as above. Define

$$\phi_{s,t}: \bigwedge^{s} F \otimes \bigwedge^{t} G \to \bigwedge^{s+t} G$$

as the map sending $f \otimes g \to \wedge^s \phi(f) \wedge g$ for $f \in \wedge^s F$ and $g \in \wedge^t G$. If we take $t = n - s$, then $\wedge^{s+t} G \cong R$; we define $I_s(\phi) = \operatorname{ann}(\operatorname{coker} \phi_{s,n-s})$, where ann denotes the annihilator in R. This ideal is nothing but the familiar "ideal of $s \times s$ minors of ϕ—the $(n - s)$th Fitting invariant of coker ϕ— and its radical is the same as the radical of $\operatorname{ann}(\operatorname{coker} \phi_{k,n-s})$

for $0 \leqslant k \leqslant n - s$. (See section 1D for a more complete statement.) Most of the computation of explicit finite free resolutions has been done for resolutions of modules of the form $\operatorname{coker}(\phi_{s,t})$, under the assumption that

$$(*) \qquad \operatorname{grade} I_{n-t} = (t+1)(m-n+t+1)$$

the largest possible value. It is known that this grade is achieved if ϕ can be represented by a matrix whose entries form an R-sequence, and thus, in particular, in the "generic" case in which ϕ is given by a matrix of indeterminates.

Finite free resolutions are known for $\operatorname{coker} \phi_{s,t}$, under the hypothesis (*), if $s = n$, $t = 0$ [8, 11], if $s = 1$, $t = 0$ [4], or if $s = n - 1$, $t = 1$, with $m = n$ or $m = n + 1$ ([14] and [23], respectively). In this paper we will construct finite free resolutions for $\operatorname{coker}(\phi_{s,0}) = \operatorname{coker}(\Lambda^s \phi)$ for all s (section 4), and we will construct some finite free complexes that look as if they should be resolutions, for $\operatorname{coker}(\phi_{s,1})$, for all s, with $m = n$ (section 6). [We have checked that these complexes are resolutions for $s = n - 1$, in which case our complex coincides with that of $[G - N]$, and for $s = 1$, in which case we get a resolution of $\Lambda^2(\operatorname{coker}(\phi))$.] All these complexes are minimal if R is local with maximal ideal M, and $\phi(F) \subset MG$.

Although the constructions of complexes in this paper are somewhat complicated, they are motivated by a rather simple philosophy, which we now describe. Generalizing the notion for ideals, an R-module M is said to be perfect if $\operatorname{grade}(\operatorname{ann} M) = pd\, M$.

It has turned out that, under hypothesis (*), $\operatorname{coker}(\phi_{s,t})$ is a perfect module for every s and t for which a free resolution is known, and it has been shown that $R/I_s(\phi) = \operatorname{coker}(\phi_{s,n-s})$ is perfect for every s [10]. It is thus natural to conjecture that $\operatorname{coker}(\phi_{s,t})$ is perfect for every s and t, so long as (*) holds. Belief in this conjecture simplifies the process of looking for a resolution of $\operatorname{coker}(\phi_{s,t})$. Both to prove the conjecture and to find a free resolution, it is enough to find a free complex $F_{s,t}(\phi)$, such that: (0) The length of $F_{s,t}(\phi)$ is $(t+1)(m-n+t+1)$; (1) $H_0(F) = \operatorname{coker}(\phi_{s,t})$; and (2) $H_i(F)$ is annihilated by some power of $I_{n-t}(\phi)$. [By Lemma 5.3 conditions (0) and (2) imply the exactness of F under hypothesis (*).]

If we drop assumption (*) and requirement (0), we get the definition of a grade-sensitive complex for the module, $\operatorname{coker}(\phi_{s,t})$. Such a complex will have the property that for any R-module M, the smallest i such that

$H^i(F, M) \neq 0$ is the length of a maximal M-sequence in $I_s(\phi)$, and for this i, $H^i(F, M) = \mathrm{Ext}^i(\mathrm{coker}\,\phi_{s,t}, M)$ [8]. Thus, grade-sensitive complexes can replace finite free resolutions for some purposes.

But our knowledge of grade-sensitive complexes is much greater than our knowledge of free resolutions; in [9] a grade-sensitive complex $F_{s,t}$ for $\mathrm{coker}(\phi_{s,t})$ is defined for every s and t $\{F_{s,t}$ is the complex called $D(\phi, *, s, n - (s + t) + 1)$ in [9]$\}$. Unfortunately, the complexes $F_{s,t}$ are infinitely long unless $t = 0$; and even if $t = 0$ they are not minimal. However, the Eagon-Northcott complex and the "generic resolution" of [4] are derived from the complexes $F_{n,0}$ and $F_{1,0}$ by a "symmetrization" process that strips away the superfluous parts of these nonminimal complexes. It thus seems reasonable to try to obtain minimal (and thus finite) free resolutions of $\mathrm{coker}(\phi_{s,t})$ under hypothesis (*), for every s and t, by somehow "paring" down the complexes $F_{s,t}$. This is essentially the program we have followed in sections 4 and 6. The Poon resolution [23] seems to be obtainable in a similar way.

To carry out this "paring" of the complexes $F_{s,t}$, we have had to introduce and study the family of functors $L_p{}^q$ from free R-modules to free R-modules that we mentioned earlier; the complexes of sections 4, 5, and 6 are expressed in terms of these functors, which are defined in section 3. They are defined as the kernels of certain natural maps between tensor products of symmetric and exterior powers and are natural building blocks for minimal resolutions since they are something like modules of "formal" cycles in some nonminimal complexes.

Throughout our work on free resolutions and in this paper in particular, we have made use of certain tools for computation in the exterior and symmetric algebras. Although the ideas here are more or less standard in the theory of bialgebras, we have looked in vain for a suitable reference. For this reason we have included, in Section 1, an introduction to multilinear algebra in the style we require.

All the material of this paper can be done, with only slight modifications, in the context of finitely generated projective modules rather than free modules. Also, it is possible to use a non-noetherian notion of grade and relax the hypothesis that R be noetherian.

Throughout this paper we will use the notations introduced above. R will denote a commutative ring, noetherian after section 3, all modules will be R-modules, and all algebras will be R-algebras. F and G will denote R-modules that will be assumed finitely generated and free after section 1; and if $\phi: F \to G$ is a homomorphism, we write $\phi_{s,t}$: $\bigwedge^s F \otimes \bigwedge^t G \to \bigwedge^{s+t} G$ for the map defined above.

1. Multilinear Algebra

In this section we will present a survey of some of the basic facts about multilinear algebra. Nearly all of the results of this section are well-known. Good general references are [1] and, for the material on divided powers, [13, §7]; [25] is a good general reference on Hopf algebras.

A) $\wedge F$ *as a Commutative, Cocommutative Bialgebra*

We denote by $\wedge F$ the exterior algebra on F— — it is the free graded commutative R-algebra generated by elements of F in degree 1. [For a graded algebra, the commutative law reads $fg = (-1)^{(\deg f)(\deg f)}gf$.] Because of the universal property of the exterior algebra, the diagonal map $F \to F \oplus F$ induces an algebra map $\wedge F \xrightarrow{\Delta} \wedge F \otimes \wedge F$. If $f \in F$ is regarded as an element of degree 1 in $\wedge F$, it may easily be checked that

$$\Delta f = f \otimes 1 + 1 \otimes f \in \bigwedge F \otimes \bigwedge F \tag{1.0}$$

The elements of degree 0 of $\wedge F$ form a ring isomorphic to R, and projection into degree 0 is an algebra map

$$\bigwedge F \xrightarrow{\epsilon} R$$

called the counit; ϵ and Δ satisfy a set of identities dual to those satisfied by the multiplication m: $\wedge F \otimes \wedge F \to \wedge F$ and unit η: $R \to \wedge F$. Thus $\wedge F$ becomes a commutative, cocommutative bialgebra. We will often write

$$\Delta g = \sum g_1{}^i \otimes g_2{}^i \qquad \text{for} \quad g \in \bigwedge F$$

and it follows from the fact that Δ is an algebra homomorphism, and from (1.0), that

$$\Delta g = 1 \otimes g + g \otimes 1 + \sum_{\substack{\deg g_1{}^i > 0 \\ \deg g_2{}^i > 0}} g_1{}^i \otimes g_2{}^i \tag{1.1}$$

For any R-module M, we will write $M^* = \mathrm{Hom}_R(M, R)$ and, if $\hat{m} \in M^*$, $m \in M$, we will write $\langle \hat{m}, m \rangle$ for the value of $\hat{m}$ on m. Since $\wedge F$ is a bialgebra, $(\wedge F)^*$ is too; for example, its multiplication is defined by the formula:

$$\langle \phi\gamma, f \rangle = \langle \phi \otimes \gamma, \Delta f \rangle \in R \qquad \text{for} \quad \phi, \gamma \in \left(\bigwedge F\right)^*, \qquad f \in \bigwedge F,$$

where $\langle \phi \otimes \gamma, f' \otimes f'' \rangle = (-1)^{(\deg \gamma)(\deg f')} \langle \phi, f' \rangle \langle \gamma, f'' \rangle$. The map $F^* \to (\wedge F)^*$ which is dual to the projection onto degree one:

$$\bigwedge F \to F$$

induces a natural algebra map

$$\bigwedge F^* \xrightarrow{\alpha} \left(\bigwedge F\right)^*$$

which is in fact a map of bialgebras.

It is easily checked that if $f_1, \ldots, f_k \in F$ and $\phi_1, \ldots, \phi_k \in F^*$ with $\langle \phi_i, f_j \rangle = \delta_{ij}$ (the Kronecker delta), then

$$\langle \alpha(\phi_1 \wedge \cdots \wedge \phi_k), f_1 \wedge \cdots \wedge f_k \rangle = (-1)^{k(k-1)/2} \tag{1.2}$$

Thus, in particular, α is an isomorphism if F is a finitely generated free or projective R-module.

A′) *SF and DF*

Similar remarks hold for the symmetric algebra $S_R(F) = S(F)$, which may be regarded as the free graded commutative algebra generated by its elements of *degree* 2 which form a module isomorphic to F. Thus, $S(F) = \sum_{k \geqslant 0} S_k(F)$ (where we have written S_k for the module of elements of degree $2k$) is a commutative algebra in the usual sense with $S_1(F) = F$; if F is a free module on n generators, $X_1 \cdots X_n$, then $S(F)$ is the polynomial ring $R[X_1, \ldots, X_n]$.

As with the exterior algebra, the diagonal map $F \to F \oplus F$ induces an algebra map

$$\Delta: S(F) \to S(F \oplus F) = S(F) \otimes S(F)$$

with

$$\Delta(f) = f \otimes 1 + 1 \otimes f \qquad \text{for} \quad f \in F,$$

and projection onto $S_0(F) = R$ gives an algebra map

$$S(F) \xrightarrow{\epsilon} R$$

Together, these make $S(F)$ into a commutative, cocommutative bialgebra.

Of course, since $S(F)$ is a bialgebra, its dual $S(F)^*$ is too. But since $S(F) = \sum_{i \geqslant 0} S_i(F)$ is an infinite sum, it is much more important to work with the so-called graded dual

$$S(F)^*_{\mathrm{gr}} = \sum_{i \geqslant 0} (S_i(F))^*$$

Once again, the module map

$$F^* = (S_1(F))^* \to S(F)^*_{\mathrm{gr}}$$

induces an algebra map $S(F^*) \xrightarrow{\alpha'} S(F)^*_{\mathrm{gr}}$. But it is *not* an isomorphism unless R contains the field of rational numbers. In fact, if $\hat{f}$ and f are elements of F^* and F, respectively, it is easy to check that

$$\langle \alpha'(\hat{f}^p), f^p \rangle = p! \langle \hat{f}, f \rangle^p$$

To get at the algebra $S(F)^*_{\mathrm{gr}}$, we define $D(F)$, the divided power algebra on F, to be the graded commutative algebra generated by elements $f^{(p)}$ (called the pth divided power of f) in degree $2p$, where $f \in F$ is regarded as an element of degree 2 in $D(F)$, satisfying:

(0) $D_0(F) = R$; $D_1(F) = F$

(1) $f^{(0)} = 1$, $f^{(1)} = f$, $f^{(i)} \in D_i(F)$, for $f \in F$

(2) $f^{(p)} f^{(q)} = \binom{p+q}{q} f^{(p+q)}$, for $f \in F$

(3) $(f+g)^{(p)} = \sum_{k=0}^{p} f^{(p-k)} g^{(k)}$, for $f, g \in F$

(4) $(fg)^{(p)} = f^p g^{(p)}$, for $f, g \in F$

(5) $f^{(p)(q)} = [p, q] f^{(pq)}$, for $f \in F$, where $[p, q] = [(pq)!]/(q! p^q!)$

[Note that, as with $S(F)$, we are writing $D_i(F)$ for the module of elements of degree $2i$.)

It is easy to see that $D(F)$ is a bialgebra and that, if F is a free R-module on generators X_i, then $D_p(F)$ is free on generators

$$\prod_i X_i^{(p_i)}$$

with $\sum p_i = p$.

Now we can define an algebra map, which we will again call α,

$$D(F^*) \xrightarrow{\alpha} S(F)^*_{\mathrm{gr}}$$

by the formula

$$\alpha(\hat{f}^{(p)}) \left(\prod_i f_i^{p_i} \right) = \begin{cases} 0 & \text{if } \sum p_i \neq p \\ \prod_i (\hat{f}(f_i))^{p_i} & \text{if } \sum p_i = p \end{cases}$$

It follows from this definition that if F is a free R-module on generators X_i, and if $\hat{X}_i \in F^*$ are the dual basis elements, then

$$\alpha(\hat{X}_i^{(p)})(X_i^p) = 1$$

So α is an isomorphism in this case. Also, if F is free, $(S(F)^*_{\mathrm{gr}})^*_{\mathrm{gr}} \cong S(F)$ as algebras, so

$$D(F^*)^*_{\mathrm{gr}} \cong S(F)$$

as algebras.

B) *Module Structures on* $\wedge F$, $S(F)$, *and* $D(F)$

We wish to consider $\wedge F$ as a $\wedge F^*$-module, and to consider $D(F^*)$ and $S(F)$ as modules over each other. To do this efficiently, we generalize:

Let A and B be graded commutative, cocommutative bialgebras with multiplication maps

$$m_A\colon A \otimes A \to A \qquad \text{and} \qquad m_B\colon B \otimes B \to B$$

and diagonal maps

$$\Delta_A\colon A \to A \otimes A \qquad \text{and} \qquad \Delta_B\colon B \to B \otimes B$$

We will always suppose, further, that A and B are connected, i.e., $A_0 = B_0 = R$.

Let $\alpha\colon B \to A^*_{\mathrm{gr}}$ be a homogeneous bialgebra homomorphism. We will show how to regard A as a B-module, and exhibit some of its nice properties.

First of all, note that if F, G, and H are R-modules, then there is a natural map

$$S_{F,G,H} = s\colon \mathrm{Hom}_R(F, G \otimes H) \to \mathrm{Hom}_R(G^* \otimes F, H)$$

defined by $s(\phi)(\gamma \otimes f) = (\gamma \otimes 1)(\phi(f)) \in R \otimes H = H$. If G is graded, then the same formula yields a map

$$s\colon \mathrm{Hom}_R(F, G \otimes H) \to \mathrm{Hom}_R(G^*_{\mathrm{gr}} \otimes F, H)$$

In particular, if $F = G = H = A$ then $s(\Delta)\colon A^*_{\mathrm{gr}} \otimes A \to A$. We define

$$n = n_B = (s(\Delta))(\alpha \otimes 1)\colon B \otimes A \to A$$

We will write $b(a)$ or ba for $n(b \otimes a)$. Note that n is easy to calculate: if $a \in A$ and $\Delta(a) = \sum a_1{}^i \otimes a_2{}^i$, then

$$b(a) = \sum \langle b, a_1{}^i \rangle a_2{}^i$$

Note that the algebra map $B \xrightarrow{\alpha} A^*_{\mathrm{gr}}$ gives rise to an algebra map $A \to A^{**}_{\mathrm{gr}} \to B^*_{\mathrm{gr}}$, so that B is also an A-module, and the following proposition can be applied to this action as well. If $a \in A$, $b \in B$, we will write $a(b) \in B$ for this action.

PROPOSITION 1.1.

(i) *The map n makes A into a B-module. This module structure is compatible with the diagonal and multiplication maps in the sense that*

(ii) $\Delta_A\colon A \to A \otimes A$ *is a map of modules over the map of rings* $m_B\colon B \otimes B \to B$, *and*

(iii) $m_A\colon A \otimes A \to A$ *is a map of modules over the map of rings* $\Delta_B\colon B \to B \otimes B$

where the $B \otimes B$-module structure on $A \otimes A$ is the graded tensor product structure. Moreover

(iv) *If bialgebras A', B' are given with a map $B' \xrightarrow{\alpha'} A'^*_{\mathrm{gr}}$, and if bialgebra maps*

$$\phi\colon A \to A' \qquad \text{and} \qquad \psi\colon B' \to B$$

are given such that

$$\langle \alpha'(b'), \phi(a) \rangle = \langle \alpha(\psi(b')), a \rangle \qquad \textit{for all} \quad a \in A, \quad b' \in B'$$

then ϕ is a map of modules over the map ψ of rings.

Remarks. If S and T are rings, $\phi\colon S \to T$ is a ring homomorphism, M is an S-module, N is a T-module, and $\psi\colon N \to M$ is a map of abelian groups, then we say that *ψ is a map of modules over ϕ* if for all $a \in S$, $b \in N$, we have

$$a\psi(b) = \psi(\phi(a)b)$$

If we write $n_{B \otimes B}$ for the structure map of the $B \otimes B$-module $A \otimes A$, (ii) is equivalent to the commutativity of the diagram

(ii′)

$$\begin{tikzcd} B \otimes B \otimes A \arrow[r, "m_B \otimes 1"] \arrow[dd, "1 \otimes 1 \otimes \Delta_A"] & B \otimes A \arrow[d, "n"] \\ & A \arrow[d, "\Delta"] \\ B \otimes B \otimes A \otimes A \arrow[r, "n_{B \otimes B}"'] & A \otimes A \end{tikzcd}$$

while (iii) expresses the commutativity of

(iii′)

$$\begin{tikzcd} B \otimes A \otimes A \arrow[r, "\Delta_B \otimes 1 \otimes 1"] \arrow[dd, "1 \otimes m_A"] & B \otimes B \otimes A \otimes A \arrow[d, "n_{B \otimes B}"] \\ & A \otimes A \arrow[d, "m_A"] \\ B \otimes A \arrow[r, "n_B"] & A \end{tikzcd}$$

Statement (iii) is often expressed by saying that the coalgebra B measures the algebra A.

A special case of statement (iii) is so familiar as to be worth special note. If $b \in \wedge F^*$ is an element of degree 1, then $\Delta(b) = 1 \otimes b + b \otimes 1$. Thus, in the situation of the proposition, if A is the exterior algebra $\wedge F$, then (iii) shows that any element of B of degree 1 acts on A as a derivation.

Statement (iv) is useful in situations like the following. If $\phi: F \to G$ is a map of R-modules, then $\wedge G$ and $\wedge F$ are $\wedge G^*$-modules, the latter via the map $\wedge \phi^*: \wedge G^* \to \wedge F^*$. Part (iv) tells us that in this setup, $\wedge \phi$ is a map of $\wedge G^*$-modules.

Proof of the Proposition. Because $\alpha: B \to A^*_{\mathrm{gr}}$ is a map of bialgebras, we may assume, for the proof, that $B = A^*_{\mathrm{gr}}$ and $\alpha = 1$.

We first establish that n makes A into a B-module. This follows readily from the cocommutativity and coassociativity of Δ_A, which we will use in the following form: If $a \in A$, we will write $\Delta a = \sum a_1{}^i \otimes a_2{}^i \in A \otimes A$ where each $a_k{}^i$ is homogeneous. Let

$$\Delta a_1{}^i = \sum a_{11}^{ij} \otimes a_{12}^{ij}$$

$$\Delta a_2{}^i = \sum a_{21}^{ij} \otimes a_{22}^{ij}$$

The coassociativity and cocommutativity of Δ assert that

$$(\#)\qquad \begin{aligned}\sum_{ij} a_{11}^{ij} \otimes a_{12}^{ij} \otimes a_2{}^i &= \sum_{ij} a_1{}^i \otimes a_{21}^{ij} \otimes a_{22}^{ij} \\ &= \sum_{ij} (-1)^{(\deg a_1{}^i)(\deg a_{21}^{ij})} a_{21}^{ij} \otimes a_1{}^i \otimes a_{22}^{ij}\end{aligned}$$

Now if b_1, b_2 are homogeneous elements of B, we have

$$\begin{aligned} b_1(b_2(a)) &= b_1\left(\sum \langle b_2 , a_1{}^i\rangle\, a_2{}^i\right) \\ &= \sum \langle b_2 , a_1{}^i\rangle\, b_1(a_2{}^i) \\ &= \sum \langle b_2 , a_1{}^i\rangle\langle b_1 , a_{21}^{ij}\rangle\, a_{22}^{ij} \\ &= \sum \langle b_1 , a_{21}^{ij}\rangle\langle b_2 , a_1{}^i\rangle\, a_{22}^{ij} \\ &= \sum (b_1 \otimes b_2 \otimes 1)((-1)^{(\deg b_2)(\deg a_{21}^{ij})} a_{21}^{ij} \otimes a_1{}^i \otimes a_{22}^{ij}) \\ &\qquad\qquad \in R \otimes R \otimes A = A \end{aligned}$$

where in the last equation we have regarded $b_1 \otimes b_2$ as being a map from $A \otimes A$ into R by the map $A^* \otimes A^* \to (A \otimes A)^*$. Using (#), and noting that all terms are zero unless $\deg a_1{}^i = \deg b_2$ we see that this last expression is equal to

$$b_1 \otimes b_2 \otimes 1 \left(\sum a_{11}^{ij} \otimes a_{12}^{ij} \otimes a_2{}^i\right)$$

On the other hand, since m_B is the dual of Δ_A under the pairing $\langle\ ,\ \rangle$, we have

$$\begin{aligned}(b_1 b_2)(a) &= \sum \langle b_1 b_2 , a_1{}^i\rangle\, a_2{}^i \\ &= \sum \langle b_1 \otimes b_2 , a_{11}^{ij} \otimes a_{12}^{ij}\rangle\, a_2{}^i \\ &= b_1 \otimes b_2 \otimes 1 \left(\sum a_{11}^{ij} \otimes a_{12}^{ij} \otimes a_2{}^i\right)\end{aligned}$$

So $b_1(b_2(a)) = (b_1 b_2)(a)$ as required.

We will next prove part (ii) by showing the commutativity of (ii′). Note that if F, F', G, G', H, H' are R-modules, and if

$$\begin{aligned}\phi&: F' \to F \\ \gamma&: G \to G' \\ \nu&: H \to H'\end{aligned}$$

then the diagram

$$\begin{tikzcd}
\mathrm{Hom}(F, G \otimes H) \arrow[r, "s_{F,G,H}"] \arrow[d, "\mathrm{Hom}(\phi,\gamma\otimes\nu)"] & \mathrm{Hom}(G^* \otimes F, H) \arrow[d, "\mathrm{Hom}(\gamma^*\otimes\phi,\nu)"] \\
\mathrm{Hom}(F', G' \otimes H') \arrow[r, "s_{F',G',H'}"'] & \mathrm{Hom}(G'^* \otimes F', H')
\end{tikzcd}$$

commutes. Applying this principle twice, we get the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A \otimes A, A \otimes A \otimes A \otimes A) \arrow[r, "s_{A\otimes A,A\otimes A,A\otimes A}"] \arrow[d, "\mathrm{Hom}(\Delta,1)"] & \mathrm{Hom}(B \otimes B \otimes A \otimes A, A \otimes A) \arrow[d, "\mathrm{Hom}(1\otimes\Delta,1)"] \\
\mathrm{Hom}(A, A \otimes A \otimes A \otimes A) \arrow[r, "s_{A,A\otimes A,A\otimes A}"] & \mathrm{Hom}(B \otimes B \otimes A, A \otimes A) \\
\mathrm{Hom}(A, A \otimes A) \arrow[r, "s_{A,A,A}"'] \arrow[u, "\mathrm{Hom}(1,\Delta\otimes\Delta)"] & \mathrm{Hom}(B \otimes A, A) \arrow[u, "\mathrm{Hom}(m\otimes 1,\Delta)"]
\end{tikzcd} \tag{**}$$

But the coassociativity and cocommutativity of the bialgebra A imply that $\Delta_{A\otimes A} \in \mathrm{Hom}(A \otimes A, A \otimes A \otimes A \otimes A)$ and $\Delta_A \in \mathrm{Hom}(A, A \otimes A)$ are sent by the vertical maps on the left-hand side of the diagram to the same map in the middle. By the commutativity of (**), the maps $s_{A\otimes A,A\otimes A,A\otimes A}(\Delta_{A\otimes A}) = n_{B\otimes B} \in \mathrm{Hom}(B \otimes B \otimes A \otimes A, A \otimes A)$ and $s_{A,A,A}(\Delta_A) = n_B \in \mathrm{Hom}(B \otimes A, A)$ are sent by the vertical maps on the right-hand side of (**) to the same map in $\mathrm{Hom}(B \otimes B \otimes A, A \otimes A)$. Thus $n_{B\otimes B}(1 \otimes \Delta) = \Delta n_B(m \otimes 1)$, which is the commutativity of (ii').

The proof of (iii') is very similar, starting from the fact that m_A is a map of coalgebras, that is, $\Delta_A m_A = m_A \otimes m_A(\Delta_{A\otimes A})$. Tracing this equality through the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(A, A \otimes A) \arrow[r, "s_{A,A,A}"] \arrow[d, "\mathrm{Hom}(m_A,1)"] & \mathrm{Hom}(B \otimes A, A) \arrow[d, "\mathrm{Hom}(1\otimes m_A,1)"] \\
\mathrm{Hom}(A \otimes A, A \otimes A) \arrow[r, "s_{A\otimes A,A,A}"] & \mathrm{Hom}(B \otimes A \otimes A, A) \\
\mathrm{Hom}(A \otimes A, A \otimes A \otimes A \otimes A) \arrow[r, "s_{A\otimes A,A\otimes A,A\otimes A}"] \arrow[u, "\mathrm{Hom}(1,m_A\otimes m_A)"] & \mathrm{Hom}(B \otimes B \otimes A \otimes A, A \otimes A) \arrow[u, "\mathrm{Hom}(\Delta_B\otimes 1\otimes 1,m_A)"]
\end{tikzcd}$$

gives $n_B(1 \otimes m_A) = m_A n_{B\otimes B}(\Delta_B \otimes 1 \otimes 1)$, the desired equality.

(iv) If $b \in B'$, $a \in A$, we must show that $\phi(\psi(b)(a)) = b\phi(a)$. But

$$\begin{aligned}\phi(\psi(b)(a)) &= \phi\left(\sum \langle \alpha'\psi(b), a_1{}^i\rangle\, a_2{}^i\right)\\ &= \sum\langle \alpha b, \phi(a_1{}^i)\rangle\, \phi(a_2{}^i)\\ &= b(\phi(a))\end{aligned}$$

since ϕ is a coalgebra homomorphism. This concludes the proof.

For clarity, we will sometimes denote the multiplication in A or B by $\wedge$. If $b \in B$, and $c, d \in A$, then the expression $c(b)(d)$ refers to the action of $c(b) \in B$ on $d \in A$.

Corollary 1.2. *Let* $b \in B$ *and let* $c, d \in A$, *with* $\deg c = 1$. *Then*

$$c(b)(d) = c \wedge (b(d)) + (-1)^{1+\deg b} b(cd)$$

Proof. By (iii),

$$b(cd) = \sum (-1)^{\deg b_2{}^i} b_1{}^i(c)\, b_2{}^i(d), \qquad \text{where} \quad \Delta b = \sum b_1{}^i \otimes b_2{}^i.$$

Since c has degree 1, the terms vanish unless $\deg b_1{}^i \leqslant 1$. If $\deg b_1{}^i = 1$, then we have

$$b_1{}^i(c) = c(b_1{}^i) \in R$$

Thus

$$\begin{aligned}b(cd) &= (-1)^{\deg b} c \wedge b(d) + \sum_{\deg b_1 = 1} (-1)^{(\deg b)-1} b_1{}^i(c) \wedge b_2{}^i(d)\\ &= (-1)^{\deg b}(c \wedge b(d) - c(b)(d))\end{aligned}$$

This gives us the desired relation.

If we assume that A is generated as an R-algebra by A_1, then Corollary 1.2 can be generalized.

Corollary 1.3. *Suppose that* A *is generated by* A_1, *and that* $b \in B$, $c, d \in A$. *Then if* $\Delta c = \sum c_1{}^i \otimes c_2{}^i$, *we have*

$$c(b)(d) = \sum (-1)^{(1+\deg b)(\deg c_2{}^i)} c_1{}^i \wedge [b(c_2{}^i \wedge d)]$$

Proof. Corollary 1.2 is the case $\deg c = 1$, since then $\Delta c = 1 \otimes c + c \otimes 1$. Using the linearity of our formula, we may assume that $c = ef$ with $\deg e = 1$. A straightforward and mildly tedious induction on $\deg c$ now completes the proof.

If F is a free module of rank n, and b is a generator for $\wedge^n F^*$, then there are isomorphisms $\wedge^k F \to \wedge^{n-k} F^*$ given by $a \to a(b)$. The following consequence of Corollary 1.3 generalizes this in a useful way:

COROLLARY 1.4. *With the hypothesis of Corollary* 1.3, *suppose further that* $A_i = 0$ *for* $i > n$, *and that* $b \in B_n$. *Then*

$$c(b)(d) = (-1)^k d(b)(c)$$

where $k = (\deg b)(1 + \deg c + \deg d) + (\deg c)(\deg d)$.

The duality in F mentioned above is given by the special case in which b and d are dual basis vectors of $\wedge^n F$ and $\wedge^n F^*$. The proof is again a straightforward induction on $\deg c$, using the fact that if $x \in A$ and $y \in B$ have the same degree, then $x(y) = y(x) \in R$.

C) *Divided Powers in* $\wedge F$

Since we have already treated the divided power algebra A, we will confine ourselves in this section to working with the bialgebra $\wedge F$. The results of this part will not be applied in this paper. The interested reader will find an application in [6].

A system of divided powers for a graded algebra A is, for each element $a \in A$ of even degree, a sequence of elements $a^{(0)}, a^{(1)}, a^{(2)} \ldots$ satisfying the following relations.

(1) $a^{(0)} = 1; \quad a^{(1)} = a; \quad \deg a^{(i)} = i \deg a$

(2) $a^{(p)} a^{(q)} = \binom{p+q}{p} a^{(p+q)}$

(3) $(a+b)^{(p)} = \sum_{k=0}^{p} a^{(p-k)} b^{(k)}$ if $\deg a = \deg b$ is even

(4) $(ab)^{(p)} = \begin{cases} 0 \text{ if } \deg a \text{ and } \deg b \text{ are odd} \\ a^p b^{(p)} \text{ if } \deg a \text{ and } \deg b \text{ are even} \end{cases}$ for $p \geqslant 2$

(5) $a^{(p)^{(q)}} = [p, q]\, a^{(pq)}$, where $[p, q] = \dfrac{(pq)!}{q!\,p^q!}$

Since elements of degree 1 in $\wedge F$ square to 0, and since $\wedge F$ is generated by its elements of degree 1, axioms 1–4 may be used to define a unique system of divided powers in $\wedge F$ as follows:

If $a \in \wedge F$ is a homogeneous element, we may write

$$a = \sum^{k} a_i$$

where each a_i is a product of elements of degree 1. Using induction on k and axioms 1–4 we obtain

$$a^{(p)} = \left(\sum_1^k a_i\right)^{(p)} = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant k} a_{i_1} \wedge \cdots \wedge a_{i_p}$$

It is easily checked that this definition also satisfies (5) so that $\bigwedge F$ has a unique system of divided powers. These are conveniently related to the diagonal of $\bigwedge F$ as follows:

LEMMA 1.5. *Let $a \in \bigwedge F$ have degree* 1, *and let $b \in \bigwedge F^*$ have even degree. Then $a(b^{(p)}) = a(b) \wedge b^{(p-1)}$.*

Proof. This can be shown by a direct computation, or by stealth, as follows. Since the system of divided powers in $\bigwedge F$ is unique and since the proposition asserts an algebraic identity, it suffices to prove the proposition in the case that the ring R is a polynomial ring over the integers and F is a free R-module. In this case axioms 1 and 2 imply $p!(b^{(p)}) = b^p$. Since a is of degree 1, it acts as a derivation on $\bigwedge F$ (formula 1.4), so $a(b^p) = pa(b) \wedge b^{p-1}$.

The proposition now follows by the torsion-freeness of $\bigwedge F^*$ over the integers.

D) *Fitting Ideals and Annihilators*

Let $\phi: F \to G$ be a map of free R-modules, and suppose rank $G = n$. We define $\phi_{s,t}$: $\bigwedge^s F \otimes \bigwedge^t G \to \bigwedge^{s+t} G$ to be the map given by multiplication: $f \otimes g \mapsto \bigwedge^s \phi(f) \wedge g$, and we define $I_s(\phi) = \operatorname{ann}(\operatorname{coker} \phi_{s,t})$. Of particular interest is the ideal $I_{s,n-s}(\phi)$; we will write simply $I_s(\phi)$ for this ideal. Since $\bigwedge^n G \cong R$, $I_s(\phi)$ is in fact nothing but the "ideal of $s \times s$ minors" of ϕ, also known as the $(n - s)$th Fitting ideal of coker(ϕ). On the other hand, if $M = \operatorname{coker} \phi$, then $\operatorname{coker}(\phi_{1,p-1}) = \bigwedge^p M$, so $I_{1,p-1}(\phi) = \operatorname{ann}(\bigwedge^p M)$. (See [1] or [7].)

PROPOSITION 1.5. *With notation as above, we have*

(1) $I(s, t) \supseteq I(s + 1, t) \supseteq (I(s, t))(I(1, t))$

if $s + t + 1 \leqslant n$. Thus in particular

$$\begin{aligned} \operatorname{Rad}(I_{1,t}(\phi)) &= \operatorname{Rad}\left(\operatorname{ann} \bigwedge^{t+1} M\right) \\ &= \operatorname{Rad}(I_{2,t}(\phi)) \ldots \\ &= \operatorname{Rad}(I_{n-t}(\phi)) \end{aligned}$$

Ons further fact that we shall need concerns the grades of the ideals $I_{s,t}$. By the above, grade $I_{s,t}(\phi) =$ grade $I_{n,t}(\phi)$.

We will also use some special cases of an important theorem of Eagon and Hochster which states:

THEOREM 1.6. *Suppose, with the above notations, that F has rank m. Then* grade $I_s(\phi) \leqslant (m-s+1)(n-s+1)$. *If, with respect to some bases of F and G the entries of a matrix for ϕ form an R-sequence, then the equality is achieved.*

We conjecture that if $m > n$, and grade $I_s(\phi) = (m-s+1)(n-s+1)$, then $I_{1,n-s}(\phi) = I_{2,n-s}(\phi) = \cdots = I_{s,n-s}(\phi)$ $(= I_s(\phi))$. This is proven, for $t = 0$, in [7].

2. DEFINITION OF THE FUNDAMENTAL MODULES

Throughout this section, F is a finitely generated free R-module. We will construct free modules $L_p{}^qF$, out of which the complexes to be considered in the rest of this paper are built. These will appear as homogeneous components of modules of cycles in a Koszul complex defined over the ring SF.

Since SF is a graded SF-module, and $\wedge F$ is a graded $\wedge F^*$-module, we may regard $SF \otimes \wedge F$ as a bigraded $SF \otimes \wedge F^*$-module. The identity map $1: F \to F$ yields, by the identification $\operatorname{Hom}(F, F) \simeq F \otimes F^*$, an element $c = c_F$ of $F \otimes F^* = S_1F \otimes \wedge^1 F^*$. We will write ∂_F: $SF \otimes \wedge F \to SF \otimes \wedge F$ for the $SF \otimes \wedge F^*$-module map given by multiplication by c; it is a map of bidegree $(2, -1)$. (Recall that S_1F is the module of elements of degree 2 in SF!) We will drop the subscript F from ∂ and c where there is no danger of confusion.

We define

$$LF = \ker \partial_F .$$

It is clearly a bigraded $SF \otimes \wedge F^*$-module and its bihomogeneous components are our fundamental building blocks. If we write

$$\partial_k{}^l: S_{k-1}F \otimes \bigwedge^{l} F \to S_k \otimes \bigwedge^{l-1} F$$

for the components of ∂_F then we have

DEFINITION. $L_p{}^qF = \ker \partial_{p+1}^{q-1}$.

The reason we have chosen this particular indexing for the bihomogeneous components of LF will become clearer from Corollary 2.3.

We first show that LF is a functor:

PROPOSITION 2.1. *Let $\phi: F \to G$ be a map of finitely generated free R-modules. The map $S\phi \otimes \wedge \phi$: $SF \otimes \wedge F \to SG \otimes \wedge G$ induces a map $L\phi: LF \to LG$, which is a map of $SF \otimes \wedge G^*$-modules.*

Proof. The algebra maps $1 \otimes \wedge \phi^*: SF \otimes \wedge G^* \to SF \otimes \wedge F^*$ and $S\phi \otimes 1: SF \otimes \wedge G^* \to SG \otimes \wedge G^*$ make both $SF \otimes \wedge F^*$ and $SG \otimes \wedge G^*$ algebras over $SF \otimes \wedge G^*$, so that both LF and LG are $SF \otimes \wedge G^*$-modules.

To see that $L\phi$ exists and is a map of $SF \otimes \wedge G^*$-modules, it suffices to show that the following diagram commutes, and consists of maps of $SF \otimes \wedge G^*$-modules:

$$\begin{tikzcd} SF \otimes \wedge F \arrow[r, "\partial_F"] \arrow[d, "S\phi\otimes\wedge\phi"'] & SF \otimes \wedge F \arrow[d, "S\phi\otimes\wedge\phi"] \\ SG \otimes \wedge G \arrow[r, "\partial_G"] & SG \otimes \wedge G \end{tikzcd} \tag{2.1}$$

Let $\phi \in \mathrm{Hom}(F, G)$ correspond to the element $c_\phi \in G \otimes F^*$, and let ∂_ϕ be the map given by multiplication by c_ϕ on $SG \otimes \wedge F$. Using this notation we can factor the two vertical maps of (2.1) as

$$\begin{tikzcd} SF \otimes \wedge F \arrow[r, "\partial_F"] \arrow[d, "S\phi\otimes 1"'] & SF \otimes \wedge F \arrow[d, "S\phi\otimes 1"] \\ SG \otimes \wedge F \arrow[r, "\partial_\phi"] \arrow[d, "1\otimes\wedge\phi"'] & SG \otimes \wedge F \arrow[d, "1\otimes\wedge\phi"] \\ SG \otimes \wedge G \arrow[r, "\partial_G"] & SG \otimes \wedge G \end{tikzcd}$$

The top square of this diagram commutes because $S\phi \otimes 1$ is an algebra homomorphism and $c_\phi = \phi \otimes 1(c_F) = S\phi \otimes 1(c_F)$. The same reasoning shows that $S\phi \otimes 1$ is a map of $SF \otimes \wedge F^*$-modules, and thus a map of $SF \otimes \wedge G^*$-modules.

The bottom square commutes because $\wedge \phi$ is a map of modules over the map $\wedge \phi^*$ of rings (Proposition 1.1, iv); and the last remark before the proof of that proposition), and $c_\phi = 1 \otimes \phi^*(c_G) = 1 \otimes \wedge \phi^*(c_G)$. Again, this also shows that $1 \otimes \wedge F$ is a map of $SG \otimes \wedge G^*$-modules, and thus a map of $SF \otimes \wedge G^*$-modules.

To analyze LF, we note first that since c is an element of odd total degree in $SF \otimes \wedge F$, we have $c^2 = 0$, and thus $\partial^2 = 0$. Consequently ∂ may be considered as the differential of a complex

$$\mathbf{A}(F)\colon \cdots \to SF \otimes \bigwedge^{k+1} F \xrightarrow{\partial} SF \otimes \bigwedge^{k} F \xrightarrow{\partial} SF \otimes \bigwedge^{k-1} F \to$$
$$\cdots \xrightarrow{\partial} SF \otimes F \xrightarrow{\partial} SF \to 0$$

If $\{x_1 \cdots x_n\}$ is a basis for F, then SF is the polynomial ring $R[x_1 \cdots x_n]$, and $\partial\colon SF \otimes F \to SF$ is the map that takes the ith basis element of $SF \otimes F = R[x_1 \cdots x_n]^n$ to the element x_i in $SF = R[x_1 ,..., x_n]$. Moreover, $SF \otimes \wedge^k F \cong \wedge^k_{SF}(SF \otimes F)$, and with these identifications, $\mathbf{A}$ is the Koszul complex resolving the ideal $(x_1 ,..., x_n) \subset R[x_1 ,..., x_n]$. Thus $\mathbf{A}$ is acyclic:

$$H_i(\mathbf{A}(F)) = \begin{cases} 0 & i > 0 \\ R & i = 0 \end{cases}$$

Decomposing $SF \otimes \wedge F$ and ∂ completely into their bihomogeneous components, we see that $\mathbf{A}(F)$ is a direct sum of complexes

$$\mathbf{A}_k(F)\colon \cdots \to S_{k-l}F \otimes \bigwedge^{l} F \xrightarrow{\partial^l_{k-l+1}} S_{k-l+1}F \otimes \bigwedge^{l-1} F \to \cdots \xrightarrow{\partial_k^1} S_kF \xrightarrow{\partial^0_{k+1}} 0.$$

The acyclicity of $\mathbf{A}$ now becomes:

Proposition 2.2. *$\mathbf{A}_k(F)$ is exact if $k > 0$, and $\mathbf{A}_0$ is the trivial complex*

$$\mathbf{A}_0(F)\colon 0 \to R \to 0.$$

This allows us to compute $L_p{}^qF$ for a number of values of p and q:

Corollary 2.3. (a) *If $p + q \neq 1$, then*

$$L_p{}^qF = \ker \partial^{q-1}_{p+1} = \operatorname{im} \partial_p{}^q = \operatorname{coker} \partial^{q+1}_{p-1}$$

(b) $L_p{}^1F = S_pF$ for all p
$L_1{}^qF = \wedge^q F$ for all $q \neq 0$
$L_p{}^0F = L_0{}^qF = 0$ for all $q \neq 1$, and all p.
$L_p{}^qF = 0$ for all $q > \operatorname{rank} F$

(c) *If* rank $F = n$, *then* $L_p{}^nF \cong S_{p-1}(F) \otimes \wedge^n F$.

Proof. Part (a) follows at once from the exactness statement of Proposition 2.1.

The various formulas in (b) came from examining the definition, and using (a). For example, $L_p{}^1F = \ker(S_pF \xrightarrow{\partial^0_{p+1}} 0)$ shows that $L_p{}^1F = S_pF$, while if $q \neq 0$, $L_1{}^q = \operatorname{coker}(0 \xrightarrow{\partial_0^{q+1}} S_0F \otimes \wedge^q F)$. The other two formulas follow similarly. Part (c) follows from the exactness of

$$0 \to S_{p-1}F \otimes \bigwedge^n F \to S_pF \otimes \bigwedge^{n-1} F \xrightarrow{\partial_{p+1}^{n-1}} S_{p+1}F \otimes \bigwedge^{n-2} F$$

Now suppose that $\phi: F \to G$ is a map of free R-modules, with

$$\operatorname{rank} F = m \geqslant \operatorname{rank} G = n.$$

The next section will describe complexes that are, under "generic circumstances," free resolutions of the modules $\operatorname{coker}(L_p{}^q\phi: L_p{}^qF \to L_p{}^qG)$. Using Corollary 2.3, we can already tell what the annihilators of these modules are.

We will write $I_p{}^q\phi$ for the ideal of R that is the annihilator of $\operatorname{coker}(L_p{}^q\phi)$. Thus $I_1{}^1 = \operatorname{ann}(\operatorname{coker}\phi)$. We retain the notation $I_k(\phi)$ for the ideal of $k \times k$ minors of ϕ.

PROPOSITION 2.4. *With the above notation,*

$$(I_1{}^1\phi)^{p+q-1} \subseteq I_p{}^q\phi \subseteq I_1{}^1\phi$$

for every p, q with $1 \leqslant p$, $1 \leqslant q \leqslant n$. In particular,

$$\operatorname{Rad}(I_p{}^q\phi) = \operatorname{Rad}(I_n\phi)$$

Proof. The first statement shows that $\operatorname{Rad}(I_p{}^q\phi) = \operatorname{Rad}(I_1{}^1\phi)$, but by Proposition 1.5, $\operatorname{Rad}(\operatorname{ann}\operatorname{coker}\phi) = \operatorname{Rad}(I_n\phi)$.

To prove the first inequality of the proposition, we first note that because $L\phi$ is a map of $SF \otimes \wedge G^*$-modules, and thus of SF-modules, we have a commutative square

$$\begin{tikzcd} F \otimes L^q_{p-1} \arrow[r, "\phi\otimes L^q_{p-1}\phi"] \arrow[d] & G \otimes L^q_{p-1}G \arrow[d] \\ L_p{}^qF \arrow[r, "L_p{}^q\phi"] & L_p{}^qG \end{tikzcd} \tag{2.2}$$

If $p - 1 \geqslant 1$, we may use Corollary 2.3a, to write $F \otimes L^q_{p-1}F$ and $L_p{}^q$ as homomorphic images:

$$F \otimes S_{p-2}F \otimes \bigwedge^q F \twoheadrightarrow F \otimes L^q_{p-1}F$$

$$S_{p-1}F \otimes \bigwedge^q F \twoheadrightarrow L_p{}^qF$$

The left-hand vertical map in diagram (2.2) is induced by the multiplication

$$F \otimes S_{p-2}F \to S_{p-1}F$$

which is an epimorphism. Thus the left-hand vertical map is an epimorphism. Of course, the same goes for the right-hand vertical map. Since $(I_1{}^1\phi)(I^q_{p-1\phi})$ clearly annihilates the cokernel of $\phi \otimes L^q_{p-1}\phi$, it annihilates coker $L_p{}^q\phi$ as well. Thus, if $p - 1 \geqslant 1$, we have

$$(I_1{}^1\phi)(I^q_{p-1}\phi) \subseteq I_p{}^q\phi$$

Iterating this inequality, we get

$$(I_1{}^1\phi)^{p-1}(I_1{}^q\phi) \subseteq I_p{}^q\phi$$

But by Proposition 1.5, $(I_1{}^1\phi)^q \subseteq I_1{}^q\phi$, so we get

$$(I_1{}^1\phi)^{p+q-1} \subseteq I_p{}^q\phi$$

as asserted.

To prove the other inequality, we begin in a similar way. Because $L\phi$ is a map of $\wedge\, G^*$-modules, the diagram

$$\begin{array}{ccc} L_p{}^qF \otimes G^* & \xrightarrow{L_p{}^q\phi \otimes 1} & L_p{}^qG \otimes G^* \\ \downarrow & & \downarrow \\ L_p^{q-1}F & \xrightarrow{L_p^{q-1}\phi} & L_p^{q-1}G \end{array} \qquad (2.3)$$

commutes. $L_p{}^qG \otimes G^*$ and $L_p^{q-1}G$, are by Corollary 2.3a, homomorphic images:

$$S_{p-1}G \otimes \bigwedge^q G \otimes G^* \twoheadrightarrow L_p{}^qG \otimes G^*$$

$$S_{p-1}G \otimes \bigwedge^{q-1} G \twoheadrightarrow L_p^{q-1}G$$

and the right-hand vertical map in diagram (2.3) is induced by the module structure map $\wedge^q G \otimes G^* \to \wedge^{q-1} G$. Thus the right-hand map in (2.3) is an epimorphism if $q \leqslant n$.

The cokernel of the top horizontal map in (2.3) is $G^* \otimes (\operatorname{coker} L_p{}^q\phi)$, which obviously has annihilator $I_p{}^q\phi$. Thus, since our analysis of diagram (2.3) shows that $G^* \otimes \operatorname{coker}(L_p{}^q\phi)$ maps onto $\operatorname{coker}(L_p^{q-1}\phi)$, we have

$$I_p{}^q\phi \subseteq I_p^{q-1}\phi$$

Iterating this inequality, we get

$$I_p{}^q\phi \subseteq I_p{}^1\phi.$$

Note next that $L_p{}^1\phi = S_p\phi$. The fact that $S_p(\phi)$ is a map of $(SG)^*$-modules [Proposition 1.1(iv)] gives us a commutative diagram

$$\begin{array}{ccc} S_pF \otimes G^* & \xrightarrow{S_p\phi\otimes 1} & S_pG \otimes G^* \\ \downarrow & & \downarrow \\ S_{p-1}F & \longrightarrow & S_{p-1}G \end{array}$$

in which, as before, the vertical right-hand map is an epimorphism. From this we conclude that

$$I_p{}^1\phi \subseteq I_{p-1}^1\phi$$

Iterating, and combining this with the inequality $I_p{}^q\phi \subseteq I_p{}^1\phi$, we get

$$I_p{}^q\phi \subseteq I_1{}^1\phi$$

as required.

Since the R-modules occurring in the complex $\mathbf{A}_k$ are all free, and since $\mathbf{A}_k$ is exact for $k > 0$, it follows that $L_p{}^qF$ is a projective module for all p, q. But in fact $L_p{}^qF$ is free.

To see this, we use the first part of the following proposition, which gives an expression for $L_p{}^q(F + R)$ in terms of $L_p{}^qF$:

PROPOSITION 2.5. (a) *There is a short exact sequence of* $S(F \oplus R) \otimes \wedge F^*$-*modules*

$$0 \to S(F \oplus R) \otimes \wedge F \to L(F \oplus R) \to LF \to 0$$

where the right-hand map is the one derived from the projection $F \oplus R \to F$, *and if we write* x *for the element* $(0, 1) \in F \oplus R$, *the left-hand map is* *multiplication by* $1 \otimes x\colon S(F \oplus R) \otimes \wedge F \to S(F \oplus R) \otimes \wedge(F \oplus R)$, *composed with the canonical map*

$$S(F \oplus R) \otimes \wedge (F \oplus R) \to L(F \oplus R)$$

given by the complex $\mathbf{A}$.

(b) *As R-modules*

$$L_p{}^q(F \oplus R) \cong S_{p-1}(F \oplus R) \otimes \bigwedge^{q-1} F \oplus L_p{}^q F$$

(c) *If F is free of* rank n, *then* $L_p{}^qF$ *is free of rank*

$$\binom{n+p-1}{q+p-1}\binom{q+p-2}{p-1}$$

Proof. Part (c) follows from part (b) using induction on the rank of F, and the standard identities:

$$\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$$

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$$

for all positive integers a, b, c.

Part (b) follows at once from part (a) since, as we have already remarked, $L_p{}^qF$ is a projective R-module.

To obtain part (a), we use the fact that the Koszul complex $\bar{\mathbf{K}}$ associated to the ideal $(x_1, \ldots, x_{n+1})$ in $R[x_1, \ldots, x_{n+1}]$ and the Koszul complex $\mathbf{K}$ associated to the ideal $(x_1, \ldots, x_n)$ in $R[x_1, \ldots, x_{n+1}]$ are related by the fact that $\bar{\mathbf{K}}$ is the mapping cylinder of the map $\mathbf{K} \to \mathbf{K}$ induced by multiplication by x_{n+1} or, as it is sometimes alternatively put, $\bar{\mathbf{K}} \cong \mathbf{K} \otimes \mathbf{X}_{n+1}$, where $\mathbf{X}_{n+1}$ is the complex $0 \to R[x_1 \cdots x_{n+1}] \xrightarrow{x_{n+1}} R[x_1 \cdots x_{n+1}] \to 0$. Clearly the cokernel of $x_{n+1}\colon \mathbf{K} \to \mathbf{K}$ is the Koszul complex associated to the ideal $(x_1, \ldots, x_n) \subset R[x_1 \cdots x_n]$. Now if $x_1, \ldots, x_n$ is a basis of F, and x_{n+1} is the basis vector $x = (0, 1) \in F \oplus R$, then we may identify $R[x_1 \cdots x_{n+1}] = S(F \oplus R)$, $\mathbf{K} = S(F \oplus R) \otimes \wedge F$, $\bar{\mathbf{K}} = S(F \oplus R) \otimes \wedge(F \oplus R)$ and $\operatorname{coker}(\mathbf{K} \xrightarrow{x_{n+1}} \mathbf{K}) = SF \otimes \wedge F = \mathbf{A}$. Thus we have an exact sequence of complexes

$$0 \to S(F \oplus R) \otimes \wedge F \xrightarrow{x_{n+1}} S(F \oplus R) \otimes \wedge (F) \to SF \otimes \wedge F \to 0$$

In fact, this is an exact sequence of differential $S(F \oplus R) \otimes \wedge F^*$-modules [where we regard SF as an $S(F \oplus R)$-module via the projection $\pi: F \oplus R \to F$].

Since $L(F \oplus R)$ is the module of cycles in the complex $S(F \oplus R) \otimes \wedge (F \oplus R)$ which is the mapping cylinder of the left-hand map, while LF is the module of cycles in $SF \otimes \wedge F$, part (a) of the proposition follows from an elementary result on mapping cylinders:

LEMMA 2.6. *Let* $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ *be a short exact sequence of differential modules and let* M *be the mapping cylinder of* f. *Then denoting the modules of cycles, in* M *and* W *by* $C(M)$ *and* $C(W)$, *respectively, we have*

$$C(M) \cong g^{-1}C(W)$$

Thus there is a short exact sequence

$$0 \to U \to C(M) \to C(W) \to 0$$

Proof. Write d_U, d_W, and d_V for the differentials in U, V, and W. As modules, $M \cong U \oplus V$, and an element $(u, v) \in M$ is a cycle if and only if $d_U(u) = 0$, and $d_V(v) = f(u)$. But the second of these conditions implies the first. For if $d_V(v) = f(u)$ then

$$fd_U(u) = d_V f(u) = d_V{}^2 v = 0$$

and f is a monomorphism, so $d_U(u) = 0$.

This proves that the following diagram is a pull-back.

$$\begin{tikzcd} C(M) \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"'] & V \arrow[d, "d_V"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

On the other hand, f is a monomorphism, the kernel of g. By a simple diagram chase, proj_2: $C(M) \to V$ is a monomorphism and is the kernel of $gd_V = d_W g$. However, $\ker d_W g = g^{-1} \ker d_W = g^{-1}C(W)$. The second statement of the lemma follows at once from the first.

The proof of Proposition 2.3(a) may easily be made to yield an actual basis of $L_p{}^q(F \oplus R)$ from one for $L_p{}^qF$. Thus we may inductively build up a basis of $L_p{}^q(F)$ itself. We will need this explicit basis for some future calculations, so we now exhibit it by exhibiting a subset of the basis of $S_{p-1}F \otimes \wedge^q F$ which will inject under ∂ to a basis of $L_p{}^qF$. Let $x_1, \ldots, x_n$ be a basis for F. If J is a subset of order q of $\{1, \ldots, n\}$, $J = \{j_1, \ldots, j_q\}$, say, we write x_J for the element $x_{j_1} \wedge \cdots \wedge x_{j_q} \in \wedge^q F$. We will identify $S_{p-1}(F)$ with the free module generated by the monomials M of degree $p-1$ in the variables $x_1, \ldots, x_n$, and we will label these monomials by the variables they involve. Thus, if $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$, we will write M_I for any monomial of the form

$$M_I = \prod_{k=1}^{s} x_{i_k}^{p_k} \qquad \text{where} \quad \sum_{k=1}^{s} p_k = p - 1, \qquad \text{and} \qquad p_k > 0 \qquad \text{for all } k$$

Also, if I, J are two subsets of $\{1, \ldots, n\}$, then we will write

$$I \leqslant J$$

if there is some element $j \in J$ which is at least as large as any element of I:

$$i \in I \Rightarrow i \leqslant j$$

With these notations we are ready to write down the basis of $L_p{}^qF$. The proof of the following result involves nothing more than an unravelling of the identifications made in the proof of Proposition 2.3.

PROPOSITION 2.7. *Let $p + q \neq 1$, and let the notations be as above. Then the elements*

$$\partial(M_I \otimes x_J) \in S_pF \otimes \bigwedge^{q-1} F, \quad \textit{where} \ M_I \otimes x_J \in S_{p-1}F \otimes \bigwedge^{q} F, \quad \textit{and} \quad I \leqslant J$$

form a basis of $L_p{}^qF$.

3. RESOLVING COKER $(L\phi)$

Given a map of finitely generated free R-modules $F \xrightarrow{\phi} G$, we have seen that there is an induced map $LF \xrightarrow{L\phi} LG$. In this section we will define some complexes that under "generic" circumstances are minimal free resolutions of $\operatorname{coker} L_p{}^q\phi$. To be more precise, suppose that

$\operatorname{rank} F = m$ and $\operatorname{rank} G = n$. Then we will define for every pair of integers p and $q \geqslant 1$ a complex

$$\mathbf{L}_p{}^q(\phi)\colon 0 \to L_p{}^m F \otimes L_{m-n}^{n-q+1} G^* \xrightarrow{d} L_p^{m-1} F \otimes L_{m-n-1}^{n-q+1} G^* \xrightarrow{d} \cdots$$
$$\xrightarrow{d} L_p^{n+1} F \otimes L_1^{n-q+1} G^* \xrightarrow{d_1} L_p{}^q F \xrightarrow{L_p{}^q(\phi)} L_p{}^q G$$

Here, and in what follows, $L_s{}^t G^*$ means $(L_s{}^t G)^*$; thus for example $L_s{}^1 G^* = (S_s G)^* \cong D_s(G^*)$, the sth component of the divided power algebra on G^*. Note that all the complexes $\mathbf{L}_p{}^q$ have length $m - n + 1$. In section 4, we will prove the following exactness criterion, which is the main theorem of this paper:

THEOREM 3.1. *Let R be a noetherian ring, and suppose that $\phi\colon F \to G$ is a map between free R-modules of ranks m and n, respectively. If* grade $I_n(\phi) = m - n + 1$, *then $\mathbf{L}_p{}^q(\phi)$ is a free resolution of* $\operatorname{coker}(L_p{}^q\phi\colon L_p{}^q F \to L_p{}^q G)$. *If, moreover, R is local with maximal ideal J, and $\phi F \subset JG$, then $\mathbf{L}_p{}^q(\phi)$ is a minimal resolution.*

Concerning the hypothesis, we note that for any $\phi\colon F \to G$ with $\operatorname{rank} F = m$, $\operatorname{rank} G = n$, we have

$$\operatorname{grade}(I_n(\phi)) \leqslant m - n + 1$$

This well-known result [8, 11] is actually implied by the theorem. Also, it is known that the equality is actually achieved, for example, where the entries in a matrix representing ϕ form an R-sequence. By Proposition 2.4, $\operatorname{Rad}(I_n(\phi)) = \operatorname{Rad}(\operatorname{ann}(\operatorname{coker} L_p{}^q\phi))$. Thus $\operatorname{grade}(I_n(\phi)) = \operatorname{grade}(\operatorname{ann}(\operatorname{coker} L_p{}^q\phi))$, so the hypothesis of the theorem really concerns the grade of the annihilator of the module being resolved. Since the length of the complex $\mathbf{L}_p{}^q\phi$ is $m - n + 1$, the theorem implies that, if grade $I_n\phi = m - n + 1$, then $\operatorname{coker} \mathbf{L}_p{}^q\phi$ is a perfect module—that is, one whose homological dimension is equal to the grade of its annihilator.

Two special cases deserve mention:

COROLLARY 3.2. *With hypothesis and notation as in the theorem, the complex $\mathbf{L}_1{}^q$ yields a resolution of* coker $\wedge^q \phi$ *for each q, $1 \leqslant q \leqslant n$, and $\mathbf{L}_p{}^1$ yields a resolution of* coker $S_p\phi$ *for each $p \geqslant 1$.*

For $p = 1$ and $q = n$ or $q = 1$, these resolutions coincide, as they must, with the Eagon-Northcott complex [11] *and the "generic resolution"* [4].

It has been known for some time [8] that powers of an ideal generated by an R-sequence $x_1, \ldots, x_t$ can be resolved by the Eagon-Northcott complex using the device of writing the power as the ideal of minors of a matrix in a simple way, for example, $(x_1, x_2, x_3)^4$ is the ideal of 4×4 minors of the 4×6 matrix:

$$\begin{pmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ x_3 & x_2 & x_1 & 0 \\ 0 & x_3 & x_2 & x_1 \\ 0 & 0 & x_3 & x_2 \\ 0 & 0 & 0 & x_3 \end{pmatrix}$$

This procedure involves choosing a set of generators for the ideal, which is often inconvenient. However, if $I \subseteq R$ is the image of a map $F \xrightarrow{\phi} R$, then I^p is the image of the map

$$S_p\phi\colon S_pF \to S_pR \cong R$$

Thus, if I is generated by an R-sequence, the corollary may be used to produce a resolution of R/I^p without recourse to a specific choice of basis.

We will now define the differentials d and d_1 of the complexes $\mathbf{L}_p{}^q$, and some more general complexes $\mathbf{L}_p^{q,r}$ which we will use in the proof of Theorem 3.1. The proof itself will be given in the next section.

We will define complexes

$$\mathbf{L}_p^{q,r}\phi\colon 0 \to L_p{}^mF \otimes L_{m-r+1}^{r-q}G^* \xrightarrow{d} L_p^{m-1}F \otimes L_{m-r}^{r-q}G^* \xrightarrow{d} \cdots$$
$$\xrightarrow{d} L_p{}^rF \otimes L_1^{r-q}G^* \xrightarrow{d_1} L_p{}^qF$$

We define the complex $\mathbf{L}_p{}^q\phi$ to be the complex $\mathbf{L}_p^{q,n+1}\phi$, augmented by the map $L_p{}^q\phi\colon L_p{}^qF \to L_p{}^qG$.

The maps d are given as follows. Since LF is an $SF \otimes \wedge F^*$-module, we may consider it, by the canonical map $\wedge F^* \cong 1 \otimes \wedge F^* \subseteq SF \otimes \wedge F^*$, as a ΛF^*-module. Similarly, $LG^* = \mathrm{Hom}_{\mathrm{gr}}(LG, R)$ is an $SG \otimes \wedge G^*$-module that we consider as an SG-module. The element

$$\phi \in \mathrm{Hom}(F, G)$$

corresponds to an element we have called c_ϕ in $F^* \otimes G$; and if we consider c_ϕ as an element of bidegree (1, 2) in $\wedge F^* \otimes SG$ by means of the identification $F^* \otimes G = \wedge F^* \otimes S_1G$, then $c_\phi{}^2 = 0$. Thus multiplication by c_ϕ induces a differential on the (quadruply) graded module $LF \otimes LG^*$ which we call d. The maps we have labeled d in the complex

$\mathbf{L}_p^{q,r}(\phi)$ are the homogeneous components of this d. More concretely put, LF is defined as a submodule of $SF \otimes \wedge G$, while LG^* (except for $L_0{}^1G^*$) may, by Corollary 2.3(a) be thought of as a submodule of $(SG \otimes \wedge G)^* \cong D(G^*) \otimes \wedge G^*$. Thus $LF \otimes LG^*$ may be thought of as a submodule of

$$LF \otimes LG^* \subseteq SF \otimes \wedge F \otimes D(G^*) \otimes \wedge G^*$$

and d is the map induced on $LF \otimes LG^*$ by action of the element $1 \otimes c_\phi \otimes 1 \in SF \otimes \wedge F^* \otimes SG \otimes \wedge G^*$ on $SF \otimes \wedge F \otimes D(G^*) \otimes \wedge G^*$.

To define the map $d_1 : L_p{}^rF \otimes L_1^{r-q}G^* \to L_p{}^qF$ note first that LF is a $\wedge G^*$-module via the map $\wedge \phi^* : \wedge G^* \to \wedge F^*$. Since $L_1^{r-q}F = 0$ for $q \geqslant r$, we may assume $q < r$ and in this case we have a canonical identification $L_1^{r-q}G = \wedge^{r-q} G$. Thus $L_1^{r-q} G^* = \wedge^{r-q} G^*$, and we define d_1 to be the structure map of the $\wedge G^*$-module LF.

LEMMA 3.3. *With the above notation, $L_p^{q,r}$ is a complex, that is*

$$d^2 = 0 \qquad \text{and} \qquad d_1 d = 0$$

Moreover, if $r = n + 1$, then $(L_p{}^q\phi)\, d_1 = 0$, so that $\mathbf{L}_p^{q,n+1}$ may be augmented by the map $L_p{}^q\phi$ to form the complex $\mathbf{L}_p{}^q\phi$.

Proof. The identity $d^2 = 0$ follows at once from the fact that $c_\phi{}^2 = 0$.

To show that $d_1 d = 0$, we note that we may assume $q < r$, and in this case the maps d and d_1 are induced by maps on $SF \otimes \wedge F \otimes D(G^*) \otimes \wedge G^*$ which we call d' and d_1', as in the following diagram

$$\begin{tikzcd}
L_p^{r+1}F \otimes L_2^{r-q}G^* \arrow[r, "d"] \arrow[d, hook] & L_p{}^rF \otimes L_1^{r-q}G^* \arrow[r, "d_1"] \arrow[d, hook] & L_p{}^qF \arrow[d, hook] \\
S_pF \otimes \bigwedge^{r} F \otimes D_1G^* \otimes \bigwedge^{r-q} G^* \arrow[r, "d'"] & S_pF \otimes \bigwedge^{r-1} F \otimes \bigwedge^{r-q} G^* \arrow[r, "d_1'"] & S_pF \otimes \bigwedge^{q-1} F
\end{tikzcd}$$

Note that $D_1G^* = G^*$, and the map d' which is given by multiplication by $1 \otimes c_\phi \otimes 1$, may also be thought of as the action of $D_1G^* = G^* = \wedge^1 G^*$ on the $\wedge G^*$-module, $\wedge F$. Similarly, d_1' is given by the action of $\wedge^{r-q}G^*$ on $\wedge^{r-1}F$. Thus, for $a \otimes b \otimes \alpha \otimes \beta \in S_pF \otimes \wedge^r F \otimes G^* \otimes \wedge^{r-q}G^*$, we have

$$\begin{aligned}
d_1'd'(a \otimes b \otimes \alpha \otimes \beta) &= d_1'(a \otimes \alpha(b) \otimes \beta) \\
&= a \otimes \beta(\alpha(b)) \\
&= a \otimes (\beta \wedge \alpha)(b)
\end{aligned}$$

But $L_2^{r-q}G^*$ is, by Corollary 2.3(a), precisely the kernel of

$$D_1G^* \otimes \bigwedge^{r-q} G^* \xrightarrow{\partial_G{}^*} \bigwedge^{r-q+1} G^*,$$

and it is easily seen that $\partial_G{}^*(\alpha \otimes \beta) = \alpha \wedge \beta$. Thus $d_1'd'$ is zero on

$$L_p^{r+1}F \otimes L_2^{r-q}G^*$$

as desired.

Finally, we take $r = n + 1$, and we wish to show that

$$(L_p{}^q\phi)\, d_1 = 0$$

Avoiding some insignificant special cases, we may assume $p + q \geqslant 2$, $q \leqslant n$, $n > 1$, in which case we can apply Corollary 2.3(a), and the identification $L_1^{n-q+1}G^* = \wedge^{n-q+1} G^*$ to produce the commutative diagram:

$$\begin{array}{ccccc}
L_p^{n+1}F \otimes L_1^{n-q+1}G^* & \xrightarrow{\quad d_1 \quad} & L_p{}^qF & \xrightarrow{\quad L_p{}^q\phi \quad} & L_p{}^qG \\
\uparrow & & \uparrow & & \uparrow \\
S_{p-1}F \otimes \bigwedge^{n+1} F \otimes \bigwedge^{n-q+1} G^* & \xrightarrow{d_1'} & S_{p-1}F \otimes \bigwedge^{q} F & \xrightarrow{S_{p-1}\phi \otimes \wedge^q\phi} & S_{p-1}G \otimes \bigwedge^{q} G
\end{array}$$

Here d_1' is again given by the action of $\wedge^{n-q+1} G^*$ on $\wedge^{n+1} F$ via $\wedge\, \phi^*$. Thus, if $a \otimes b \otimes \beta \in S_{p-1}F \otimes \wedge^{n+1} F$, we have

$$\left(S_{p-1}\phi \otimes \bigwedge^{q} \phi\right) d_1'(a \otimes b \otimes \beta) = S_{p-1}\phi(a) \otimes \bigwedge^{q} \phi(\beta(b))$$

Now $\wedge\, \phi$ is a map of $\wedge\, G^*$-modules, so

$$\bigwedge^{q} \phi(\beta(b)) = \beta\left(\bigwedge^{n+1} \phi(b)\right)$$

but $\wedge^{n+1} \phi(b) \in \wedge^{n+1} G^* = 0$, since rank $G = n$. Thus $(L_p{}^q\phi)\, d_1 = 0$.

Remark. If $r > n + 1$, it is still possible to augment the complex $L_p^{q,r}(\phi)$ is an interesting way. For such r there are complexes

$$0 \longrightarrow L_p{}^m F \otimes L_{m-r+1}^{r-q} G^* \xrightarrow{d} \cdots \xrightarrow{d} L_p{}^r F \otimes L_1^{r-q} G^* \xrightarrow{d_1} L_p{}^q F$$

$$\xrightarrow{e_1} L_p^{r-(n+1)} F \otimes L_1^{n-r+q+1} G \longrightarrow L_p^{r-(n-1)-1} F \otimes L_2^{n-r+q+1} G$$

$$\longrightarrow \cdots \longrightarrow L_p{}^1 F \otimes L_{r-(n+1)}^{n-r+q+1} G \longrightarrow L_{p+r-(n+1)}^{n-r+q+1} G$$

which generalize the complexes used by Lebelt [20], and whose exactness properties seem to be parallel to those of the $\mathbf{L}_p{}^q$.

4. Proof of the Main Theorem

We wish to show that, under the hypothesis that grade $I_n(\phi) = m - n + 1$, the complex $\mathbf{L}_p{}^q(\phi)$ of Theorem 3.1 is exact and that, if $\phi F \subset JG$, then $\mathbf{L}_p{}^q$ is a minimal complex.

The second of these statements follows at once from the definition of the maps d, d_1, and $L_p{}^q\phi$ that make up the complex $\mathbf{L}_p{}^q\phi$, for if $\phi = 0$, then the maps d, d_1, and $L_p{}^q\phi$ are all 0, and the definition of these maps clearly commutes with reduction modulo J (or any other "base change").

To prove the exactness of the complexes $\mathbf{L}_p{}^q\phi$ we will employ the exactness criterion of Peskine-Szpiro (which we state in form slightly weaker than the original):

Lemma 4.1. ("Lemme d'acyclicité", [21, 22]). *Let R be a noetherian local ring with maximal ideal J, and let*

$$\mathbf{L}: 0 \to L_k \to L_{k-1} \to \cdots \to L_1 \to L_0$$

be a complex of free R-modules. The associated primes P of the homology $H(\mathbf{L})$ all satisfy grade $P < k$.

This seems always to be applied by means of the following corollary (which can also be deduced from the main theorem of [3]):

Corollary 4.2. *Let R be a noetherian ring, and let*

$$\mathbf{L}: 0 \to L_k \to \cdots \to L_1 \to L_0$$

be a complex of finitely generated free R-modules. If for every prime ideal $P \subset R$ with grade $P < k$ *the localized complex* $(\mathbf{L})_P$ *is exact, then* $\mathbf{L}$ *is exact.*

Proof. If $H(\mathbf{L}) \neq 0$, then $H(\mathbf{L})$ must have some associated prime P. By the lemma, grade $P < k$; but then $0 \neq H(\mathbf{L})_P = H(\mathbf{L}_P)$, contradicting the hypothesis.

Now the complexes $\mathbf{L}_p{}^q(\phi)$ may all be seen to have length $m - n + 1$, so, by the corollary, we need only prove the exactness after localizing at an arbitrary prime P of grade $< m - n + 1$. Since grade $I_n(\phi) = m - n + 1$ by hypothesis, we will have $(I_n\phi)_P = I_n(\phi_P) = R_P$. But this implies that $\phi: F \to G$ is a split epimorphism. Thus, it suffices to prove the following proposition:

PROPOSITION 4.3. *Let R be any ring and $\phi: F \to G$ be a split epimorphism of free modules, with* rank $F = m$, rank $G = n$. *Then the complexes*

$$L_p{}^q(\phi)$$

are exact for all $p \geqslant 1$, $1 \leqslant q \leqslant n$.

Remark. We have already shown that coker $L_p{}^q\phi$ is annihilated by some power of $I_n\phi$, so that if ϕ is a split epimorphism, $L_p{}^q\phi$ is a split epimorphism as well. Thus, we will really show that under the hypothesis of the proposition, $\mathbf{L}_p{}^q\phi$ is split exact. To do this it is of course sufficient to prove that for any maximal ideal P of R, the complex

$$R/P \otimes \mathbf{L}_p{}^q\phi$$

is exact. Since the formation of $L_p{}^q\phi$ "commutes with base change" we now see that it would actually be enough to prove Proposition 4.3 in the case where R is a field. But this does not seem to simplify the proof.

Proof of Proposition 4.3. We will prove the proposition by examining a double complex $\mathbf{C}$ in which the complex $\mathbf{L}_p{}^q\phi$, without the terms $L_p{}^qF$ and $L_p{}^qG$, appears as one of the columns. The other columns are complexes of the form $(\mathbf{L}_p^{q+j,q+j+1}\phi) \otimes \wedge^j G^*$. Lemma 4.4 exhibits conditions on $\mathbf{C}$ under which we can deduce Proposition 4.3, and Lemma 4.5 shows the validity of these conditions by analyzing the homology of complexes of the form $\underline{L}_p^{t,t+1}\phi$. (If $p + t + 1 > n$, this homology is actually nonzero!)

Consider the following double complex:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & \vdots & & \\
 & & & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & & \longrightarrow & L_p^{n+3}F \otimes L_3^{n-q+1}G^* & \to & 0 \\
 & & & & & & & & & & \vdots & & \downarrow & & \\
 & & & & & & & & & & & \longrightarrow & L_p^{n+2}F \otimes L_2^{n-q+1}G^* & \to & 0 \\
 & & & & & & & & & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & & & & & L_p^{n+1}F \otimes S_1G^* \otimes \bigwedge^{n-q} G^* & \to & L_p^{n+1}F \otimes L_1^{n-q+1}G^* & \to & 0 \\
 & & \downarrow & & \downarrow & & & & & & {\scriptstyle d}\downarrow & & \downarrow & & \\
n-q & 0 \to & L_p{}^nF \otimes S_{n-q}G^* & \to & L_p{}^nF \otimes S_{n-q}G^* \otimes \bigwedge^1 G^* & \to & & & \cdots & \xrightarrow{1\otimes\partial_{G^*}} & L_p{}^nF \otimes \bigwedge^{n-q} G^* & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & & & \downarrow & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & \iddots & & & & & \\
2 & 0 \to & L_p^{q+2}F \otimes S_2G^* & \xrightarrow{1\otimes\partial_{G^*}} & L_p^{q+2}F \otimes S_1G^* \otimes \bigwedge^1 G^* & \to & L_p^{q+2}F \otimes \bigwedge^2 G^* & \to & 0 & & & & & & \\
 & & {\scriptstyle d}\downarrow & & {\scriptstyle d_1\otimes 1}\downarrow & & \downarrow & & & & & & & & \\
1 & 0 \to & L_p^{q+1}F \otimes S_1G^* & \xrightarrow{1\otimes\partial_{G^*}} & L_p^{q+1}F \otimes \bigwedge^1 G^* & \longrightarrow & 0 & & & & & & & & \\
 & & {\scriptstyle d_1}\downarrow & & \downarrow & & & & & & & & & & \\
0 & 0 \longrightarrow & L_p{}^qF & \longrightarrow & 0 & & & & & & & & & & \\
 & & 0 & & & & & & & & & & & & \\
 & & 0 & & 1 & & 2 & & \cdots & & n-q & & n-q+1 & &
\end{array}
$$

Here the term in row i and column j, for $i \leqslant n - q$, is $L_p^{q+i}F \otimes S_iG^* \otimes \Lambda^j G^*$, while the term in row i and column $n - q + 1$ is $L_p^{q+i}F \otimes L_{i-n}^{n-q+1}G^*$ for $i > n$ and 0 otherwise. The boundary maps are taken so that the ith row, for $i \leqslant n - q$, is (in the notation of section 2) the complex

$$L_p^{q+i}F \otimes (\mathbf{A}_i(G))^*$$

while for $i > n - q$ they are the same complexes, with the $n - q + 1$st free module replaced by $L_p^{q+i}F \otimes L_{i-n+q}^{i-q}G^*$. Since the complexes $\mathbf{A}_iG$ are exact for $i \geqslant 1$, and L_{i-n+q}^{i-q} is the $(n - q)$th module of cycles of $\mathbf{A}_i$, we see that every row of $\mathbf{C}$ except for the 0th is exact. The homology of the 0th row is of course $L_p{}^qF$. In such circumstances an easy spectral sequence argument or a mildly tedious diagram chase yields:

LEMMA 4.4. *Let* $\mathbf{C}$ *be as above. Suppose that, for* $i \leqslant n - q$, *the* ith *column of* $\mathbf{C}$ *has a possibly nonvanishing homology module only in the* $(n - q)$th *row; call this homology group* T_i. *Suppose further that the horizontal differentials* $1 \otimes \partial_G{}^*$ *of* $\mathbf{C}$ *induce maps* $T_i \to T_{i+1}$ *that make the sequence*

$$\mathbf{I}\colon 0 \to T_0 \to T_1 \to \cdots \to T_{n-q-1} \to T_{n-q}$$

exact, with $\operatorname{coker}(T_{n-q-1} \to T_{n-q}) = L_p{}^qG$. *Then the* $(n - q + 1)$st *column of* $\mathbf{C}$ *is acyclic, and may be augmented* (*with maps arising from the spectral sequence*) *to an exact sequence*

$$\cdots \to L_p^{n+2}F \otimes L_2^{n-q+1}G^* \xrightarrow{d} L_p^{n+1}F \otimes L_1^{n-q+1}G^* \xrightarrow{\pm d_1} L_p{}^qF \to L_p{}^qG \to 0$$

Remark. If we only wanted to prove the proposition in the special case $p = 1$, we could have used a somewhat simpler result on double complexes; for in this case it will turn out that the columns of $\mathbf{C}$ are acyclic (Lemma 4.5).

Proof. We give the spectral sequence proof, leaving the diagram-chase version to the interested reader. Since all but the 0th row of $\mathbf{C}$ is exact and that row has only one term, one of the two spectral sequences associated with $\mathbf{C}$ collapses at the first (E^1) term, and the homology of the "total complex" of $\mathbf{C}$ is the nonvanishing term in the 0th row, $L_p{}^qF$. On the other hand, the hypotheses of the lemma imply that the other spectral sequence of $\mathbf{C}$ collapses at the second term; thus there is an induced filtration of the total homology $L_p{}^qF$ in which the homology of the last column maps monomorphically into $L_p{}^qF$, and the cokernel is the homology $L_p{}^qG$ of $\mathbf{T}$. Since only the 0th total homology module is nonzero, the last column of $\mathbf{C}$ must be exact except possibly at $L_p^{n+1}F \otimes L_1^{n-q+1}G^*$. The map from the homology of the last column into $L_p{}^qF$ is given by a map

$$L_p^{n+1}F \otimes L_1^{n-q+1}G^* \to L_p{}^qF$$

which comes from alternately "pulling back" and "pushing down" along the sequence of maps along the lower edge of the double complex $\mathbf{C}$; we wish to show it is (plus or minus) the map d_1. To be precise, if we make the usual identifications $L_1{}^iG^* = G^*$ and $S_1G^* = G^*$, then this sequence of maps is

$$\begin{tikzcd}
 & & {} \arrow[d, dotted, "1\otimes\partial_{G^*}"'] & L_p^{n+1}F \otimes G^* \otimes \bigwedge^{n-q} G^* \arrow[r, two heads, "1\otimes\partial_{G^*}"] \arrow[d, "d_1\otimes 1"] & L_p^{n+1}F \otimes \bigwedge^{n-q+1} G^* \\
 & & \cdots \arrow[r, two heads] \arrow[d, "d_1\otimes 1"'] & L_p{}^nF \otimes \bigwedge^{n-q} G^* & \\
 & \cdots \arrow[r, two heads] & {} \arrow[d, "d_1\otimes 1"] & & \\
L_p^{q+1}F \otimes G^* \arrow[r, two heads, "1"] \arrow[d, "d_1"] & L_p^{q+1}F \otimes G^* & & & \\
L_p{}^qF & & & &
\end{tikzcd}$$

Thus, if we take $a \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-q+1} \in L_p^{n+1} \otimes \Lambda^{n-q+1} G^*$, its image in $L_p{}^qF$ can be expressed, using the $\Lambda\, G^*$-module structure of $L_p{}^qF$, as

$$\alpha_{n-q+1}(\alpha_{n-q}(\cdots(\alpha_1(a))\cdots) = (\alpha_{n-q+1} \wedge \cdots \wedge \alpha_1)(a) = \pm d_1(\alpha \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-q+1}).$$

The conclusion of the lemma follows.

Proposition 4.3 will follow at once from Lemma 4.4 once we know that the hypotheses of Lemma 4.4 are valid. It is not even necessary to check that the map

$$L_p{}^qF \to L_p{}^qG$$

in the conclusion of Lemma 4.4 is the map $L_p{}^q\phi$! For we know by Proposition 2.4 that, in the circumstances of Proposition 4.3, $L_p{}^q\phi$ induces an epimorphism of coker d_1 onto $L_p{}^qG$. But by Lemma 4.4

$\operatorname{coker}(d_1) \simeq L_p{}^q G$; so d_1 must induce an isomorphism ($\operatorname{coker} d_1$) $\to L_p{}^q G$ (otherwise the kernel would be a summand, violating the additivity of ranks of free modules). Consequently, $L_p{}^q\phi$ is exact as claimed in the proposition.

It remains to verify that the hypotheses of Lemma 4.4 are satisfied for **C**. First of all, we must compute the homology of the columns of **C**. As we have said, the jth column of **C** is the complex

$$(\mathbf{L}_p^{q+j,q+j+1}\phi) \otimes \bigwedge^j G^*.$$

Its homology is plainly the homology of

$$\mathbf{L}_p^{q+j,q+j+1}\phi\colon \cdots \to L_p^{n+1}F \otimes S_{n-q-j+1}G^* \to L_p{}^n F \otimes S_{n-q-j}G^*$$
$$\to L_p^{n-1}L \otimes S_{n-q-j-1}G^* \to \cdots \to L_p^{q+j+1}F \otimes G^* \to L_p^{q+j}F$$

tensored with $\wedge^j G^*$.

To compute this, let $\tau_k \in S_kG \otimes S_kG^*$ be the image of $1 \in R$ under the map that is dual to the pairing

$$\langle\ ,\ \rangle\colon S_kG^* \otimes S_kG \to R$$

τ_k is a sort of generalized trace element. We now use the SG-module structure on LG by letting

$$\bar{x}_k\colon L_p{}^nG = L_p{}^nG \otimes S_0G^* \to L_{p+k}^n G \otimes S_kG^*$$

be the map induced by multiplication by τ_k. Finally, let $\sigma\colon G \to F$ be a splitting of $\phi\colon F \to G$, and let

$$x_k\colon L_p{}^nG \to L_{p+k}^n F \otimes S_kG^*$$

be the composite

$$L_p{}^nG \xrightarrow{\bar{x}_k} L_{p+k}^n G \otimes S_kG^* \xrightarrow{L_{p+k}^n\sigma\otimes 1} L_{p+k}^n F \otimes S_kG^*$$

The map x_k actually identifies $L_p{}^nG$ with the homology of $\mathbf{L}_{p+k}^{n-k,n-k+1}\phi$:

Lemma 4.5. *For each p, q with $1 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, the map x_{n-q} induces an isomorphism*

$$L_{p+q-n}^n G \simeq H(\mathbf{L}_p^{q,q+1}\phi)$$

In particular, the homology of the jth *column of* $\mathbf{C}$ *occurs, for* $j \leqslant n - q$, *only in the* $(n - q)$th *row of* $\mathbf{C}$.

Proof. First suppose that $\phi: F \to G$ is an isomorphism, i.e., that $\operatorname{rank} F = n$, and let $\sigma: G \to F$ be its inverse. In this case $L_p^{n+i}F = 0$ for $i > 0$, so the complex we are interested in takes the form

$$\mathbf{L}_p^{q,q+1}\phi\colon 0 \to L_p{}^nF \otimes S_{n-q}G^* \to L_p^{n-1}F \otimes S_{n-q-1}G^*$$
$$\to \cdots \to L_p^{q+1}F \otimes G^* \to L_p{}^qF$$

We will establish the lemma, in this case, by induction on p. First, let $p = 1$. We will use the identification $L_1{}^tF \simeq \wedge^t F$.

Let $f \in \wedge^n F^*$ be a generator. Then the map

$$a \mapsto a(f)\colon \bigwedge^{t} F \to \bigwedge^{n-t} F^*$$

is an isomorphism, and if $\alpha \in F^*$, then by Corollary 1.2,

$$[\alpha(a)](f) = \alpha \wedge a(f) \pm a(\alpha \wedge f) = \alpha \wedge a(f)$$

since $\alpha \wedge f \in \wedge^{n+1} F^* = 0$. Thus $a \to a(f)$ is an isomorphism of $\wedge F^*$-modules. But this implies that there is an isomorphism of complexes:

$$\begin{array}{ccccccc}
\mathbf{L}^{q,q+1}\phi\colon 0 \longrightarrow & \bigwedge^{n} F \otimes S_{n-q}G^* & \longrightarrow \cdots \longrightarrow & \bigwedge^{q+1} F \otimes G^* & \longrightarrow & \bigwedge^{q} F & \longrightarrow 0 \\
 & \Vert\wr & & \Vert\wr & & \Vert\wr & \\
0 \to & \bigwedge^{0} F^* \otimes S_{n-q}G^* & \to \cdots \to & \bigwedge^{n-q+1} F^* \otimes G^* & \to & \bigwedge^{n-q} F^* & \to 0 \\
 & \Vert\wr\ \wedge^0\sigma^*\times 1 & & \wedge^{n-q-1}\sigma^*\times 1\ \Vert\wr & & \Vert\wr\ \wedge^{n-q}\sigma^* & \\
(\mathbf{A}_{n-q}G)^*\colon 0 \to & \bigwedge^{0} G^* \otimes S_{n-q}G^* & \to \cdots \to & \bigwedge^{n-q-1} G^* \otimes G^* & \to & \bigwedge^{n-q} G^* & \to 0
\end{array}$$

where $\mathbf{A}_{n-q}G$ is the complex described in section 2. By Proposition 2.2, $\mathbf{A}_{n-q}G$ is exact, and thus split exact, so that $(\mathbf{A}_{n-q}G)^*$ and $\mathbf{L}_1^{q,q+1}$ are exact, unless $q = n$, in which case the complex $L_1^{q,q+1}\phi$ degenerates to

$$0 \to L_1{}^nF \to 0$$

On the other hand, $L_{1+q-n}^{n}G = 0$ for $q < n$ and the map $x_0 : L_1{}^nG \to L_1{}^nF = \bigwedge^n F$ is just the isomorphism $L_1{}^n\sigma$. This proves Lemma 4.5 in case $\phi : F \cong G$ and $p = 1$.

We now let $p > 1$ be arbitrary, while still supposing that $\phi : F \to G$ is an isomorphism with inverse σ. The short exact sequences of $SF \otimes \bigwedge F^*$-modules

$$0 \to L_{p-1}^{t+1}F \xrightarrow{\text{inclusion}} S_{p-1}F \otimes \bigwedge^{t} F \xrightarrow{\partial_F} L_p{}^tF \to 0$$

with which we are provided by Corollary 2.3(a), gives rise to a short exact sequence of complexes

$$\begin{tikzcd}
\mathbf{L}_{p-1}^{q+1,q+2}\phi : 0 \arrow[rr] \arrow[d, hook] & & L_{p-1}^{n}F \otimes S_{n-q-1}G^* \arrow[r] \arrow[d] & \cdots \arrow[r] & L_{p-1}^{q+1}F \arrow[d] \\
S_{p-1}F \otimes \mathbf{L}_1^{q,q+1}\phi : 0 \arrow[r] \arrow[d] & S_{p-1}F \otimes \bigwedge^{n} F \times S_{n-q}G^* \arrow[r] & S_{p-1}F \otimes \bigwedge^{n-1} F \otimes S_{n-q-1}G^* \arrow[r] \arrow[d, "\partial_F \otimes 1"] & \cdots \arrow[r] & S_{p-1}F \otimes \bigwedge^{q} F \arrow[d] \\
\mathbf{L}_p^{q,q+1}\phi : 0 \arrow[r] & L_p{}^nF \otimes S_{n-q}G^* \arrow[r] & L_p^{n-1}F \otimes S_{n-q-1}G^* \arrow[r] & \cdots \arrow[r] & L_p{}^qF
\end{tikzcd} \tag{4.1}$$

If $q = n$, this diagram degenerates to

$$\begin{tikzcd}
S_{p-1}F \otimes \mathbf{L}_1^{n,n+1}\phi : 0 \arrow[r] \arrow[d, "\wr"] & S_{p-1}F \otimes \bigwedge^{n} F \otimes S_0G^* \arrow[r] \arrow[d, "\wr"] & 0 \\
\mathbf{L}_p^{q,q+1}\phi : 0 \arrow[r] & L_p{}^nF \otimes S_0G^* \arrow[r] & 0
\end{tikzcd}$$

and the homology is isomorphic to $L_p{}^nG$ by $x_0 = L_p{}^n\sigma$, as before.

If $q < n$, then the complex $S_{p-1}F \otimes \mathbf{L}_1^{q,q+1}\phi$ is exact by the case $p = 1$ of our lemma, so our induction on p, together with the long exact sequence in homology, associated to (4.1) gives

$$L_{p+q-n}^{n}G \overset{x_{n-q-1}}{\cong} H(\mathbf{L}_{p-1}^{q+1,q+2}\phi) \overset{\xi}{\cong} H(L_p^{q,q+1}\phi),$$

where ξ is the connecting homomorphism associated to (4.1). We wish to show that $\xi x_{n-q-1} = x_{n-q}$. Since ξ is induced by "pushing forward and pulling back" along the maps of the diagram

$$\begin{array}{ccc} & & L^n_{p-1}F \otimes S_{n-q-1}G^* \\ & & \downarrow \\ S_{p-1}F \otimes \bigwedge^{n} F \otimes S_{n-q}G^* & \to & S_{p-1}F \otimes \bigwedge^{n-1} F \otimes S_{n-q-1}G^* \\ \downarrow & & \\ L_p{}^nF \otimes S_{n-q}G^* & & \end{array}$$

in (4.1), it suffices to show that the diagram

$$\begin{tikzcd}[column sep=small]
 & & L^n_{p-1}F \otimes S_{n-q-1}G^* \arrow[d, "\text{inclusion}\otimes 1"] \\
L^n_{p+q-n}G \arrow[urr, bend left, "x_{n-q-1}"] \arrow[r, "(\partial_F\otimes 1)^{-1}x_{n-q}"] \arrow[dr, bend right, "x_{n-q}"'] & S_{p-1}F \otimes \bigwedge^{n} F \otimes S_{n-q}G^* \arrow[r, "1\otimes d"] \arrow[d, "\partial_F\otimes 1", shift left] \arrow[d, equal, shift right] & S_{p-1}F \otimes \bigwedge^{n-1} F \otimes S_{n-q-1}G^* \\
 & L_p{}^nF \otimes S_{n-q}G^* &
\end{tikzcd}$$

commutes.

Identifying $L_p{}^nF$ and $L^n_{p+q-n}G$ with $S_{p-1}F \otimes \wedge^n F$ and $S_{p+q-n-1}G \otimes \wedge^n G$ by the isomorphisms ∂_F and ∂_G, respectively, and identifying F and G by the isomorphism ϕ, it suffices to show the commutativity of

$$\begin{array}{ccc} S_{p+q-n}G \otimes \bigwedge^{n} G & \xrightarrow{\bar{x}_{n-q-1}} & S_{p-2}G \otimes \bigwedge^{n} G \otimes S_{n-q-1}G^* \\ \bar{x}_{n-q}\downarrow & & \downarrow \partial_F \otimes 1 \\ S_{p-1}G \otimes \bigwedge^{n} G \otimes S_{n-q}G^* & \xrightarrow{1\otimes d} & S_{p-1}G \otimes \bigwedge^{n-1} F \otimes S_{n-q-1}G^* \end{array}$$

where d is the differential of the complex $\mathbf{L}_1^{q,q+1}(1)$. Choosing a basis and a dual basis e_i and ϵ_i for G and G^*, we must prove that, for

$$a \otimes b \otimes 1 \in S_{p+q-n}G \otimes \bigwedge^{n} G \otimes S_0G^*,$$

$$\sum_i (1 \otimes \epsilon_i \otimes e_i)\, \tau_{n-q}(a \otimes b \otimes 1) = \sum_i (e_i \otimes \epsilon_i \otimes 1)\, \tau_{n-q-1}(a \otimes b \otimes 1)$$

or, what comes to the same thing,

$$(1 \otimes e_i)(\tau_{n-q}) = (e_i \otimes 1)\, \tau_{n-q-1} \in S_{n-q}G \otimes S_{n-q-1}G^*$$

To do this, we will use bases (although there is a conceptual proof: if we regard $SG \otimes SG^* = SG \otimes D(G^*)$ as $D_{SG}(SG \otimes G^*)$, then τ_k is the kth divided power of $\tau_1 = \sum e_i \otimes \epsilon_i$. Thus by Lemma 1.5,

$$\begin{aligned}(1 \otimes e_i)(\tau_{n-q}) &= (1 \otimes e_i)(\tau_1) \cdot \tau_{n-q-1} \\ &= (e_i \otimes 1) \cdot \tau_{n-q-1}).\end{aligned}$$

Let u be a multiindex of length n; that is $u = (u_1 ,..., u_n)$, where the u_i are positive integers, and set $|u| = \sum u_i$, $e^u = \prod_i e_i^{u_i} \in SG$, $\epsilon^{(u)} = \prod \epsilon_i^{(u_i)} \in D(G^*) = SG^*$. Then $\tau_k = \sum_{|u|=k} e^u \otimes \epsilon^{(u)}$ and we have

$$\begin{aligned}(1 \otimes e_i)(\tau_{n-q}) &= \sum_{|u|=n-q} e^u \otimes e_i(\epsilon^{(u)}) \\ &= \sum_{\substack{|u|=n-q \\ u_i \neq 0}} e^u \otimes e_i(\epsilon^{(u)})\end{aligned}$$

since $e_i(\epsilon^{(u)}) = 0$ if $u_i = 0$. Writing $u' = (u_1 ,..., u_{i-1}, u_i - 1, u_{i+1} ,..., u_n)$ we have

$$\begin{aligned}\sum_{\substack{|u|=n-q \\ u_i \neq 0}} e^u \otimes e_i(\epsilon^{(u)}) &= \sum_{\substack{|u|=n-q \\ u_i \neq 0}} e_i e^{u'} \otimes \epsilon^{(u')} \\ &= \sum_{|u|=n-q-1} e_i e^u \otimes \epsilon^{(u)} \\ &= (e_i \otimes 1) \cdot \tau_{n-q-1}\end{aligned}$$

as required. This concludes the proof of the lemma for the case in which $\operatorname{rank} F = \operatorname{rank} G$.

For the general case of the lemma, we use induction on the rank of F. Since $\phi: F \to G$ is onto, we have $F = G \oplus \ker \phi$, and we may assume that $\ker \phi$ is free (for example by localizing).

Thus, if $\operatorname{rank} F \neq \operatorname{rank} G$, we may write $F = F' \oplus R$, $\phi = (\phi', 0)$: $F' \oplus R \to G$, with $\sigma': G \to F'$ the splitting of $\phi': F' \to G$ induced by $\sigma: G \to F$. Write $\pi: F \to F'$ for the projection; then since $\phi = \phi'\pi$, the element of $F' \otimes G$ corresponding to ϕ' satisfies $(\pi^* \otimes 1)\, c_{\phi'} = c_\phi$.

Thus the action of $c_{\phi'}$ on $SF \otimes \wedge G^*$ or $LF \otimes \wedge G^*$ is the same as that of $c_{\phi'}$ and the short exact sequence of $S(F' \oplus R) \otimes \wedge F'^*$-modules

$$0 \to S(F' \oplus R) \otimes \bigwedge F' \to L(F' \oplus R) \to LF' \to 0$$

with which Proposition 2.5 provides us gives rise to a short exact sequence of complexes,

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
S_{p-1}F \otimes \mathbf{L}_1^{q-1,q}\phi': & \cdots \longrightarrow & S_{p-1}F \otimes \bigwedge^{n-1} F' \otimes S_{n-q}G^* & \longrightarrow \cdots & \\
\downarrow & & \downarrow & & \\
\mathbf{L}_p^{q,q+1}\phi: & \cdots \longrightarrow & L_p{}^n(F + R) \otimes S_{n-q}G^* & \longrightarrow \cdots & \\
\downarrow & & \downarrow & & \\
\mathbf{L}_p^{q,q+1}\phi': & \cdots \longrightarrow & L_p{}^nF \otimes S_{n-q}G^* & \longrightarrow \cdots & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

But we already know that, for $q \leqslant n$, $H(\mathbf{L}_1^{q-1,q}) = 0$; so the natural projection

$$\mathbf{L}_p^{q,q+1}\phi \to \mathbf{L}_p^{q,q+1}\phi'$$

induces an isomorphism on homology. To prove the lemma it now suffices to observe that, since $\phi = (\text{inclusion})\,\phi'$, the diagram

$$\begin{tikzcd}
 & L_p{}^n(F \oplus R) \otimes S_{n-q}G^* \arrow[dd, "L_p{}^n(\pi)\otimes 1"] \\
L^n_{p+q-n}G \arrow[ur, "x_{n-q}"] \arrow[dr, "x_{n-q}"'] & \\
 & L_p{}^n(F) \otimes S_{n-q}G^*
\end{tikzcd}$$

commutes.

In view of Lemma 4.5, the homology of the jth column of $\mathbf{C}$ occurs in the $(n - q)$th row and is

$$L^n_{p+q+j-n}G \otimes \bigwedge^j G^*$$

so the sequence $\mathbf{T}$ of Lemma 4.4 has the form:

$$\mathbf{T}\colon 0 \to L^n_{p+q-n}G \to L^n_{p+q-n+1}G \otimes G^* \to \cdots$$

$$\to L^n_{p-1}G \otimes \bigwedge^{n-q-1} G^* \to L_p{}^nG \otimes \bigwedge^{n-q} G^*$$

We will first show that the differential in $\mathbf{T}$ is multiplication by the element $c = c_G \in G \otimes G^* = S_1G \otimes \Lambda^1 G^*$ corresponding to the identity map $G \to G$. We will denote this multiplication map by $\bar{\partial}$. The differential in $\mathbf{T}$ is, by definition, induced by the differential $1 \otimes \partial^*$ of the $(n - q)$th row of $\mathbf{C}$ where

$$\partial\colon SG \otimes \Lambda G \to SG \otimes \Lambda G$$

is the map induced by multiplication by the same element c. Thus we must prove that

$$\begin{array}{ccc}
L^n_{p+q+j-n}G \otimes \bigwedge^j G^* & \xrightarrow{\quad\bar{\partial}\quad} & L^n_{p+q+j+1-n}G \otimes \bigwedge^{j+1} G^* \\
\Big\downarrow{\scriptstyle x_{n-q-j}\otimes 1} & & \Big\downarrow{\scriptstyle x_{n-q-j-1}\otimes 1} \\
L_p{}^nF \otimes S_{n-q-j}G^* \otimes \bigwedge^j G^* & \xrightarrow{1\otimes\partial^*} & L_p{}^nF \otimes S_{n-q-j-1}G^* \otimes \bigwedge^{j+1} G^*
\end{array}$$

commutes.

Clearly

$$\begin{array}{ccc}
L_p{}^nG \otimes S_{n-q-j}G^* \otimes \bigwedge^j G^* & \xrightarrow{1\otimes\partial^*} & L_p{}^nG \otimes S_{n-q-j-1}G^* \otimes \bigwedge^{j+1} G^* \\
\Big\downarrow{\scriptstyle L_p{}^n\sigma\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle L_p{}^n\sigma\otimes 1\otimes 1} \\
L_p{}^nF \otimes S_{n-q-j}G^* \otimes \bigwedge^j G^* & \xrightarrow{1\otimes\partial^*} & L_p{}^nF \otimes S_{n-q-j-1}G^* \otimes \bigwedge^{j+1} G^*
\end{array}$$

commutes, so it is enough to prove that

$$(1 \otimes \partial^*)\, \bar{x}_{n-q-j} = \bar{x}_{n-q-j-1}\bar{\partial}$$

Also, we may identify $L_p{}^nG$ with $S_{p-1}G \otimes \wedge^n G$ for every p. Thus, it suffices to show

$$\begin{array}{ccc} S_{p+q+j-n-1}G \otimes \bigwedge^n G \otimes \bigwedge^j G^* & \xrightarrow{\bar{\partial}} & S_{p+q+j-n}G \otimes \bigwedge^n G \otimes \bigwedge^{j+1} G^* \\ \Big\downarrow {\scriptstyle \bar{x}_{n-q-j}\otimes 1} & & \Big\downarrow {\scriptstyle \bar{x}_{n-q-j-1}\otimes 1} \\ S_{p-1}G \otimes \bigwedge^n G \otimes S_{n-q-j}G^* \times \bigwedge^j G^* & \xrightarrow{1\otimes 1\otimes \partial^*} & S_{p-1}G \otimes \bigwedge^n G \otimes S_{n-q-j-1}G^* \otimes \bigwedge^{j+1} G^* \end{array}$$

commutes.

In terms of elements, let $e_1, \ldots, e_n$ and $\epsilon_1, \ldots, \epsilon_n$ be a basis and dual basis for G and G^*. We will use the same multi-index notation as in the proof of Lemma 4.5, for elements of SG and SG^*.

For

$$a \otimes b \otimes \alpha \in S_{p+q+j-n-1}G \otimes \bigwedge^n G \otimes \bigwedge^j G^*$$

we have

$$\begin{aligned} &1 \otimes 1 \otimes \partial^*(\bar{x}_{n-q-j} \otimes 1(a \otimes b \otimes \alpha)) \\ &\quad = \sum_i \sum_{|u|=n-q-j} e^u a \otimes b \otimes e_i(\epsilon^{(u)}) \otimes \epsilon_i \wedge \alpha \end{aligned}$$

while

$$\begin{aligned} &\bar{x}_{n-q-j-1} \otimes 1(\bar{\partial}(a \otimes b \otimes \alpha)) \\ &\quad = \sum_{|u|=n-q-j-1} \sum_i e^u e_i a \otimes b \otimes \epsilon^{(u)} \otimes \epsilon_i \wedge \alpha \end{aligned}$$

Interchanging the order of summation in the upper sum and then suppressing the terms for which $u_i = 0$, we see at once exactly as in the proof of Lemma 4.5, that these two sums are equal.

We now know that

$$\bar{\partial}: L_{p+q-n+1}G \otimes \bigwedge^j G^* \to L_{p+q-n+j+1}G \otimes \bigwedge^{j+1} G^*$$

is the differential of the complex $\mathbf{T}$. Identifying $L^n_{p+q-n+j}G$ with $S_{p+q-n+j}G \otimes \wedge^n G$ as usual, we may map

$$L^n_{p+q-n+j}G \otimes \bigwedge^j G^* = S_{p+q-n+j-1}G \otimes \bigwedge^n G \otimes \bigwedge^j G^* \to S_{p+q-n+j-1}G \otimes \bigwedge^{n-j} G$$

by sending an element $a \otimes b \otimes \alpha \in S_{p+q-n+j-1}G \otimes \wedge^n G \otimes \wedge^j G^*$ to $a \otimes \alpha(b)$. By virtue of Corollary 1.2, this yields an isomorphism of complexes between $\mathbf{T}$ and the complex $\mathbf{T}'$: $0 \to S_{p+q-n-1}G \otimes \wedge^n G \to$

$S_{p+q-n-2}G \otimes \wedge^{n-1} G \to \cdots \to S_{p-1}G \otimes \wedge^q G$ which is the complex $\mathbf{A}_{p+q-1}G$, truncated at $S_{p-1}G \otimes \wedge^q G$. By Proposition 2.2 and Proposition 2.3(a), this is exact, and $\operatorname{coker}(S_{p-2}G \otimes \wedge^{q+1} G \to S_{p-1}G \otimes \wedge^q G) = L_p{}^qG$, precisely as required for the hypothesis of Lemma 4.4. This concludes the proof of Theorem 3.1.

5. An Application: Some Generically Perfect Ideals

Let F and G be free R-modules of ranks m and n, respectively, with $m \geqslant n$, and let $\phi: F \to G$ and $a: R \to F$ be maps. Set $b = \phi a: R \to G$. In this section we will consider the ideal $J(\phi, a)$ generated by $I_n(\phi)$ and $\operatorname{im} b^*$. Our work was inspired by some work of Herzog [16] in which he shows that, for $m = n$ or $m = n + 1$, the ideals $J(\phi, a)$ are generically perfect of height m. He does this by exhibiting their minimal free resolutions. We will show, more generally, that for any $m \geqslant n$ a complex $K(\phi, a)$ of length m can be constructed from the complexes $\mathbf{L}_1{}^q(\phi)$, which will be a free resolution of $J(\phi, a)$ in case grade $J(\phi, a) \geqslant m$. For the generic case (i.e., when the entries x_{ij} of a matrix for ϕ, and the entire $y_1, \ldots, y_m$ of a matrix for a are all independent indeterminates), we do have grade $J(\phi, a) = m$, and thus the ideals $J(\phi, a)$ are "generically perfect."

We now define the complex $K(\phi, a)$. For each $q \geqslant 1$ the following diagram commutes:

$$(*) \qquad \begin{array}{ccc} \bigwedge^{n-q+1} F & \xrightarrow{\wedge^{n-q+1}\phi} & \bigwedge^{n-q+1} G \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ \bigwedge^{n-q+2} F & \xrightarrow[\wedge^{n-q+2}\phi]{} & \bigwedge^{n-q+2} G \end{array}$$

where the vertical maps are multiplication in $\wedge F$ and $\wedge G$ by the elements $a(1) \in \wedge^1 F$ and $b(1) \in \wedge^1 G$, which we again call a and b, respectively. We will define maps of complexes

$$\begin{array}{c} \mathbf{L}_1^{n-q+1}\phi \\ \downarrow{\scriptstyle u^q} \\ \mathbf{L}_1^{n-q+2}\phi \end{array}$$

which are given by (*) in degrees 1 and 2, and which make the complexes $\mathbf{L}_1^{n-q+1}(\phi)$ into the rows of a double complex:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & L^n_{m-n}G^* \otimes \bigwedge^m F \arrow[r] \arrow[d] & \cdots \arrow[r] & L_2{}^nG^* \otimes \bigwedge^{n+2} F \arrow[r] \arrow[d] & L_1{}^nG^* \otimes \bigwedge^{n+1} F \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[d] \\
(**) & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & L^q_{m-n}G^* \otimes \bigwedge^m F \arrow[r] \arrow[d, "u^q_{m-n}"] & \cdots \arrow[r] & L_2{}^qG^* \otimes \bigwedge^{n+2} F \arrow[r] \arrow[d, "u_2{}^q"] & L_1{}^qG^* \otimes \bigwedge^{n+1} F \arrow[r] \arrow[d, "u_1{}^q"] & \bigwedge^{n-q+1} F \arrow[r] \arrow[d, "a"] & \bigwedge^{n-q+1} G \arrow[d, "b"] \\
0 \arrow[r] & L^{q-1}_{m-n}G^* \otimes \bigwedge^m F \arrow[r] \arrow[d] & \cdots \arrow[r] & L_2^{q-1}G^* \otimes \bigwedge^{n+2} F \arrow[r] \arrow[d] & L_1^{q-1}G^* \otimes \bigwedge^{n+1} G \arrow[r] \arrow[d] & \bigwedge^{n-q+2} F \arrow[r] \arrow[d] & \bigwedge^{n-q+2} G \arrow[d] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & L^1_{m-n}G^* \otimes \bigwedge^m F \arrow[r] & \cdots \arrow[r] & L_2{}^1G^* \otimes \bigwedge^{n+2} F \arrow[r] & L_1{}^1G^* \otimes \bigwedge^{n+1} F \arrow[r] & \bigwedge^n F \arrow[r] & \bigwedge^n G
\end{tikzcd}$$

We define $\mathbf{K}(\phi, a)$ to be the total complex associated to the double complex (**).

The remainder of this section will be devoted to the definitions of the vertical maps $u_p{}^q$ in the doubly complex (**) and the proof of the following results.

THEOREM 5.1. *Let $F \xrightarrow{\phi} G$ be a map of free R-modules with* rank $F = m \geqslant$ rank $G = n$. *Let $a: R \to F$ and let $\mathbf{K}(\phi, a)$ be the total complex associated to the double complex* (**).

Let $J(\phi, a)$ be the ideal of R generated by the images of

$$\psi: \bigwedge^n G^* \otimes \bigwedge^n F \to R \qquad \text{and} \qquad b^*: G^* \to R$$

where $b = \phi a$. Then

(1) The homology of $\mathbf{K}(\phi, a)$ is annihilated by $J(\phi, a)$.

(2) If R is a local ring with maximal ideal M, and if $\phi(F) \subseteq MG$ and $a \in MF$ then $\mathbf{K}(\phi, a)$ is a minimal complex.

(3) Grade $J(\phi, a) \leqslant m$, and $\mathbf{K}(\phi, a)$ is exact if and only if grade $J(\phi, a) = m$, in which case $J(\phi, a)$ is perfect.

In the generic case these conditions are actually satisfied; more precisely we have

THEOREM 5.2. *Let S be any commutative regular noetherian ring and let*

$$R = S[x_{ij}, y_k]_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m}$$

be the polynomial ring in $mn + m$ indeterminates with $m \geqslant n$. Let $\phi\colon R^m \to R^n$ be the map with matrix (x_{ij}), and let $a\colon R \to R^m$ be the map with matrix $(y_1, \ldots, y_m)$. Then $J(\phi, a)$ is a perfect ideal of grade m. *Further, if $n \geqslant 2$, then $\mathbf{K}(\phi, a)$ is the minimal free resolution of $R/J(\phi, a)$, and the type of the Macaulay ring $R/J(\phi, a)$ is given by the binomial coefficient $\binom{m-2}{n-1}$.*

Recall that if $J \subseteq R$ is a perfect ideal of grade m in a regular ring R, then the type of R/J is by definition the minimal number of generators of $\operatorname{Ext}^m(R/J, R)$. It is an invariant of R/J [17].

The proofs of Theorems 5.1 and 5.2 depend only on the existence of the maps $\mathbf{L}_1^{n-q+1}(\phi) \to^{u^q} \mathbf{L}_1^{n-q+2}(\phi)$ making (**) into a double complex and on the fact that $I(a) = \operatorname{Im}(a^*\colon F^* \to R)$, then $u^q(\mathbf{L}_1^{n-q+1}(\phi)) \subseteq I(a)\mathbf{L}_1^{n-q+2}(\phi)$. They do *not* depend on the precise form of the maps u^q, which we nevertheless must give in order to establish their existence.

To define the maps $u_p{}^q\colon L_p{}^qG^* \otimes \wedge^{n+p} F \to L_p^{q-1}G^* \otimes \wedge^{n+p} F$, we will find a convenient basis for $L_p{}^qG^*$ and make certain conventions. It will then be necessary to interpret the boundary maps of the complexes $\mathbf{L}_1^{n-q+1}(\phi)$ in terms of these conventions.

Let $\{g_1, \ldots, g_n\}$ and $\{\gamma_1, \ldots, \gamma_n\}$ be dual bases of G and G^*, respectively. In Proposition 2.7 it was shown that a basis for $L_p{}^qG$ consisted of the elements $\{\partial(M_I \otimes g_J) \mid I \geqslant J\}$ where $\partial\colon S_{p-1}G \otimes \wedge^q G \to S_pG \otimes \wedge^{q-1} G$, M_I is a monomial of degree $p - 1$ in the variables g_i with $i \in I$ (that is $M_I = \prod_{i \in I} g_i^{\nu_i}$) and $g_J = g_{j_1} \wedge \cdots \wedge g_{j_q}$ where $J = \{j_1, \ldots, j_q\}$. Since $L_p{}^qG^*$ is the image of $\partial^*\colon S_pG \otimes \wedge^{q-1} G^* \to S_{p-1}G^* \otimes \wedge^q G^*$, we will describe elements of $S_pG^* \otimes \wedge^{q-1} G^*$ whose images under ∂^* yield a basis for $L_p{}^qG^*$. The proof that these do actually yield a basis is exactly parallel to that of Proposition 2.7, so we omit it.

If J is a subset of $\{1,..., n\}$ and $M_J' = \prod_{j\in J} g_j^{\nu_j}$ is a monomial of degree p in S_pG, denote by M_J the element $\prod_{j\in j} \gamma_j^{(\nu_j)} \in D_p(G^*) = S_pG^*$. (Note that the subscript J in M_J does not determine M_J—it merely tells which basis elements γ_j occur.) The elements $\{M_J\}$ then are the dual bases to the monomial basis of S_pG. Similarly, if $I = \{i_1 ,..., i_{q-1}\} \subset \{1,..., n\}$ with $i_1 < \cdots < i_{q-1}$, denote by γ_I the element

$$\gamma_i \wedge \cdots \wedge y_{i_{q-1}} \in \bigwedge^{q-1} (G^*) = \bigwedge^{q-1} G^*.$$

The elements $\{M_J \otimes \gamma_I\}$ form a basis of $S_pG^* \otimes \wedge^{q-1} G^*$. We will say that $J > I$ if some element of J is strictly greater than every element of I, and we will say the $M_J \otimes \gamma_I$ is *distinguished* if $J > I$. If $J > I$, then $J = J_1 \cup K$ with $J_1 \cap K = \phi$, $J_1 \leqslant I$ (i.e., every element of J_1 is less than or equal to some element of I), and every element of K is greater than every element of I. It then is clear that $M_J = M_{J_1} M_K$, where the multiplication is that of the algebra SG^*, and that M_{J_1} and M_K are uniquely determined by M_j and I. The element M_K is in S_tG^* for some $t \geqslant 1$ and the distinguished basis element $M_J \otimes \gamma_I$ is called *amply distinguished* if $t \geqslant 2$. We now state without proof:

PROPOSITION 5.3. *The elements* $\{\partial^*(M_J \otimes \gamma_I)/M_J \otimes \gamma_I$ *is distinguished*$\}$ *form a basis for* $L_p{}^qG^*$.

Using this notation, we can make explicit the boundary map $d: L_p{}^qG^* \otimes \wedge^{n+p} F \to L_{p-1}^q G^* \otimes \wedge^{n+p-1} F$ in the complex $\mathbf{L}_1^{n-q+1}(\phi)$:

PROPOSITION 5.4. *Let* $\phi: F \to G$ *be a map of free modules, and let* $\{g_i\}$ *and* $\{\gamma_i\}$ *be dual bases of* G *and* G^*. *With the foregoing terminology and notation, if* $M_J \otimes \gamma_I = M_{J_1} M_K \otimes \gamma_I$ *is amply distinguished, then*

$$d(\partial^*(M_J \otimes \gamma_I) \otimes f) = \sum_{j\in J} \partial^*(g_j(M_J) \otimes \gamma_I) \otimes \gamma_j(f)$$

where $f \in \wedge^{n+p} F$. *If* $M_J \otimes \gamma_I = M_{J_1} M_K \otimes \gamma_I$ *is not amply distinguished, then there exists a unique* $k \in K$ *and* $M_K = \gamma_k$. *If* $i = i_{q-1}$ *is the largest element of* I, *then*

$$d(\partial^*(M_J \otimes \gamma_I) \otimes f) = \sum_{j\in J_1} \partial^*(g_j(M_J) \otimes \gamma_I) \otimes \gamma_j(f)$$

$$+ \sum_{j\in J_1 - I} \partial^*(g_j(M_J) \gamma_i \otimes \gamma_{I \cup \{j\} - \{i\}}) \otimes \gamma_k(f)$$

Proof. If $M_J \otimes \gamma_I$ is amply distinguished, then $g_i(M_J) \otimes \gamma_I$ is a distinguished basis element of $S_{p-1}G^* \otimes \Lambda^{q-1} G^*$ so that the first part of the proposition is merely an explicit restatement of the definition of the boundary map d in $\mathbf{L}_1^{n-q+1}(\phi)$. If $M_J \otimes \gamma_I$ is merely distinguished, then

$$d(\partial^*(M_{J_1}\gamma_k \otimes \gamma_I) \otimes f) = \sum_{j \in J_1} \partial^*(g_j(M_{J_1})\gamma_k \otimes \gamma_I) \otimes \gamma_j(f)$$

$$+ \partial^*(M_{J_1} \otimes \gamma_I) \otimes \gamma_k(f)$$

Now the elements $g_j(M_{J_1})\gamma_k \otimes \gamma_I$ are distinguished for $j \in J_1$, so the only problem arises with the undistinguished element $M_{J_1} \otimes \gamma_I$. However, the element $M_{J_1}\gamma_i \otimes \gamma_{I-\{i\}}$ in $S_pG^* \otimes \Lambda^{q-2} G^*$ is such that $\partial^*(M_{J_1}\gamma_i \otimes \gamma_{I-\{i\}}) = M_{J_1} \otimes \gamma_I + \sum \pm g(M_{J_1})\gamma_i \otimes \gamma_{I\cup\{j\}-\{i\}}$ so that

$$\partial^*(M_{J_1} \otimes \gamma_I) = \sum_{j \in J_1 - I} \pm \partial^*(g_j(M_{J_1})\gamma_i \otimes \gamma_{I\cup\{j\}-\{i\}})$$

and we see that the elements $g_j(M_{J_1})\gamma_i \otimes \gamma_{I\cup\{j\}-\{i\}}$ are distinguished. Thus

$$d(\partial^*(M_{J_1}\gamma_k \otimes \gamma_I) \otimes f) = \sum_{j \in J_1} \pm \partial^*(g_j(M_{J_1})\gamma_k \otimes \gamma_I) \otimes \gamma_j(f)$$

$$+ \sum_{j \in J_1 - I} \pm \partial^*(g_j(M_{J_1})\gamma_i \otimes \gamma_{I\cup\{j\}-\{i\}}) \otimes \gamma_k(f)$$

We now will define the maps

$$u_p{}^q : L_p{}^qG^* \otimes \bigwedge^{n+p} F \to L_p^{q-1}G^* \otimes \bigwedge^{n+p} F$$

of the double complex (**) associated to maps $\phi: F \to G$ and $a: R \to F$. If $M_J \otimes \gamma_I$ is an amply distinguished element, and $f \in \Lambda^{n+p} F$, we set

$$u_p{}^q(M_J \otimes \gamma_I \otimes f) = M_J \otimes a(\gamma_I) \otimes f$$

where $a(\gamma_I)$ is the action of the element $a = a(1) \in \Lambda^1 F$ on $\gamma_I \in \Lambda\, G^*$. If $M_J \otimes \gamma_I$ is not amply distinguished, we let i be the greatest element of I and write $M_J = M_{J_1}\gamma_k$ with $k > i$, and every element of $J_1 \leqslant i$. In this case we define

$$u_p{}^q(M_J \otimes \gamma_I \otimes f) = M_J \otimes a(\gamma_I) \otimes f + M_{J_1}\gamma_i \otimes \gamma_{I-\{i\}} \otimes a \wedge \gamma_k(f)$$

To check that this definition of the maps $u_p{}^q$ makes(**) into a double complex is long but straightforward work, so we will only sketch a method which the interested reader may apply if he wishes. It is necessary to check the commutativity of the diagram

$$\begin{array}{ccc} L_p{}^qG^* \otimes \overset{n+p}{\otimes} F & \longrightarrow & L_{p-1}^qG^* \otimes \overset{n+p-1}{\bigwedge} F \\ \downarrow{\scriptstyle u_p{}^q} & & \downarrow{\scriptstyle u_{p-i}^q} \\ L_p^{q-1}G^* \otimes \overset{n-p}{\bigwedge} F & \longrightarrow & L_{p-1}^{q-1}G^* \otimes \overset{n+p-1}{\bigwedge} F \end{array}$$

and the fact that $u_p^{q-1}u_p{}^q = 0$. One does both these things by applying the maps to basis elements $M_j \otimes \gamma_I \otimes f$ of $L_p{}^qG^* \otimes \Lambda^{n+p} F$. If $M_J \otimes \gamma_I$ is amply distinguished, there is no great difficulty. In case $M_J \otimes \gamma_I$ is not amply distinguished the computation is facilitated by considering three cases:

(1) Every element of J is $\leqslant$ the greatest element of $I - \{i\}$

(2) $i \in J$

(3) $i \notin J$ but some element of J is $>$ the greatest element of $I - \{i\}$, where we have as usual written i for the greatest element in I.

This completes our description of the double complex (**).

We now turn to the proofs of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Part (2) of Theorem 5.1 follows at once from our construction of (**). We will now show that Parts (1) and (3) follow easily from Theorem 5.2. (See [19] for a general discussion of this phenomenon.) Let $S = Z$, the ring of integers, and $R_0 = S[X_{ij}, Y_k]$. Let $\phi_0: R_0{}^m \to R_0{}^n$ be given by the matrix (X_{ij}), and let $a_0: R \to R^m$ be given by the matrix $(Y_1, ..., Y_m)$, as in Theorem 5.2. Then we know that $\mathbf{K}(\phi_0, a_0)$ is the minimal free resolution of $R_0/J(\phi_0, a_0)$. If R, ϕ, and a are as in Theorem 5.1, then $\mathbf{K}(\phi, a)$ is a specialization of $\mathbf{K}(\phi_0, a_0)$; that is, there is a unique homomorphism $R_0{}^\zeta \to R$ such that $\mathbf{K}(\phi, a) = R \otimes_{R_0} \mathbf{K}(\phi_0, a_0)$. If $r \in J(\phi_0, a_0)$, then the map

$$\mathbf{K}(\phi_0, a_0) \xrightarrow{r} \mathbf{K}(\phi_0, a_0)$$

induced by multiplication by r, induces 0 on $R_0/J(\phi_0, a_0)$, and thus is homotopic to 0 by some homotopy s. But then $R \otimes s$ is a homotopy on $\mathbf{K}(\phi, a)$ which shows that multiplication by $\zeta(r)$ is homotopic to zero on $\mathbf{K}(\phi, a)$. Thus $\zeta(r)$ annihilates the homology of $\mathbf{K}(\phi, a)$. Since $J(\phi, a) = R\zeta(J(\phi_0, a_0))$, part (1) is proven.

As for part (3), we make use of Lemma 4.1 which shows that if the homology of $\mathbf{K}(\phi, a)$ is annihilated by an ideal of grade $\geqslant m$, then $\mathbf{K}(\phi, a)$ is exact.

Thus, if grade $J(\phi, a) \geqslant m$ then $\mathbf{K}(\phi, a)$ is exact, and $pd(R/J(\phi, a)) \leqslant m$. However, for any ideal J we have

$$\text{grade } J \leqslant ht J \leqslant pd\,(R/J)$$

so this forces grade $J(\phi, a) = m$.

Proof of Theorem 5.2. If R, ϕ, and a are as in Theorem 5.2, then it is well-known that

$$\text{grade } I_n(\phi) = m - n + 1$$

(see, e.g., [11]). Thus we may apply the following lemma, which is a weak version of Theorem 5.1, 1).

LEMMA 5.5. *In the setup of Theorem* 5.1, *suppose that* grade $I_n(\phi) = m - n + 1$. *Then some power of the ideal* $J = J(\phi, a)$ *annihilates the homology of* $\mathbf{K}(\phi, a)$.

Proof. We must of course give a proof independent of Theorem 5.2! By Theorem 3.1, the rows of (**) are exact under the hypotheses of the lemma. By the spectral sequence of the double complex (**), the homology of $\mathbf{K}(\phi, a)$ is the same as the homology of the complex of cokernels of the maps $\wedge^p \phi$, for $p = 1,\ldots, n$. That is, if we let $C_p = $ coker $\wedge^{n-p} \phi$, the maps in the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\phi"] & \bigwedge^2 F \arrow[r] \arrow[d, "\wedge^2\phi"] & \cdots \arrow[r] & \bigwedge^{n-p} F \arrow[r] \arrow[d, "\wedge^{n-p}\phi"] & \cdots \arrow[r] & \bigwedge^n F \arrow[d, "\wedge^n\phi"] \\
G \arrow[r] \arrow[d] & \bigwedge^2 G \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigwedge^{n-p} G \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigwedge^n G \arrow[d] \\
C_{n-1} \arrow[d] & C_{n-2} \arrow[d] & & C_p \arrow[d] & & C_0 \arrow[d] \\
0 & 0 & & 0 & & 0
\end{tikzcd}$$

induce maps $C_p \to C_{p+1}$ which make the C_p into a complex $\mathbf{C}$ having the same homology as $\mathbf{K}(\phi, a)$. By Proposition 1.5 the C_p are themselves annihilated by $I_n(\phi)$, so the same goes for the homology of $\mathbf{K}(\phi, a)$. Thus we need only show that some power of $I(b)$, the ideal generated by the coefficients of b, kills $H(\mathbf{K}(\phi, a))$. To this end, let $b = (b_1, ..., b_n) \in R^n = G$, and let T be the localization of R at the multiplicatively closed set generated by one of the b_i; we must show that

$$T \otimes \mathbf{K}(\phi, a)$$

is exact. But b and a are basis elements of $T \otimes F$ and $T \otimes G$, respectively, so that the first two rows of the following diagram are Koszul complexes associated with the unit ideal.

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "a"] \arrow[d, equal] & T \otimes F \arrow[r, "a"] \arrow[d, "T\otimes\phi"] & \bigwedge^2 (T \otimes F) \arrow[r, "a"] \arrow[d, "\bigwedge^2(T\otimes\phi)"] & \cdots \arrow[r, "a"] & \bigwedge^{n-p}(T \otimes F) \arrow[r, "a"] \arrow[d, "\bigwedge^{n-p}(T\otimes\phi)"] & \cdots \arrow[r, "a"] & \bigwedge^n (T \otimes F) \arrow[d, "\bigwedge^n(T\otimes\phi)"] & \\
0 \arrow[r] & T \arrow[r, "b"] \arrow[d] & T \otimes G \arrow[r, "b"] \arrow[d] & \bigwedge^2 (T \otimes G) \arrow[r, "b"] \arrow[d] & \cdots \arrow[r] & \bigwedge^{n-p}(T \otimes G) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigwedge^n (T \otimes G) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & T \otimes C_{n-1} \arrow[r] \arrow[d] & T \otimes C_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & T \otimes C_p \arrow[r] \arrow[d] & \cdots \arrow[r] & T \otimes C_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 &
\end{tikzcd}$$

Thus they are split exact and have "compatible" splittings given by action on the $\wedge(T \otimes F^*)$-modules $\wedge(T \otimes F)$ and $\wedge(T \otimes G)$ by a dual basis element to a in $\wedge^1(T \otimes F^*)$. This shows that $T \otimes \mathbf{C}$ is split exact, so that $T \otimes H(\mathbf{C}) = 0$. Thus some power of b_i annihilates $H(\mathbf{K}(\phi, a))$. Since i was arbitrary, the lemma is proven.

To finish the proof the Theorem 5.2 it is enough to show that with ϕ and a as in Theorem 5.2, $\mathbf{K}(\phi, a)$ is exact. By Lemmas 4.1 and 5.5, it suffices to show that, in this case, grade $J(\phi, a) \geqslant m$.

LEMMA 5.6. *Let ϕ, a be as in Theorem* 5.2. *Then* grade $J(\phi, a) \geqslant m$.

Proof. We proceed by induction on n, the case $n = 1$ being trivial. Assume that grade $J(\phi, a) < m$. Then there is a prime ideal P containing $J(\phi, a)$ such that $\text{grade}_{R_P} PR_P < m$. Define ϕ': $R^m \to R^{n-1}$ to be the map whose matrix is (X_{ij}) with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n - 1$. By our induction assumption, grade $J(\phi', a) \geqslant m$, so that P cannot contain $J(\phi', a)$. However, if $b = \phi a$ and $b' = \phi' a$, it is clear that $I(b) \supseteq I(b')$ [where we write $I(b)$ for im b^*, and $I(b')$ for im b'^*]. Thus, P must fail to contain $I_{n-1}(\phi')$; we may clearly assume that P does not contain the $n-1 \times n-1$ minor

$$\begin{vmatrix} X_{11} & \cdots & X_{1,n-1} \\ \vdots & & \vdots \\ X_{n-1,1} & \cdots & X_{n-1,n-1} \end{vmatrix}$$

Let $T = R_P$. In T, the above minor is invertible, so that the ideal in T generated by $I(b')$ is equal to that generated by $Y_1, \ldots, Y_{n-1}$.

Now $T/(Y_1, \ldots, Y_{n-1})$ is a localization of $R[X_{ij}, Y_n, \ldots, Y_m]$, and $\phi \otimes T/(Y_1, \ldots, Y_{n-1})$ is a localization of a "generic" $m \times n$ matrix. Thus

$$\operatorname{grade} \frac{(I_n(\phi), Y_1, \ldots, Y_{n-1})}{(Y_1, \ldots, Y_{n-1})} = m - n + 1$$

Since $Y_1, \ldots, Y_{n-1}$ is a T-sequence, this shows that, in T,

$$\operatorname{grade}(J(\phi, a)) \geqslant \operatorname{grade}(I_n(\phi), I(b')) \geqslant m$$

which contradicts the assumptions

$$TJ(T, a) \subseteq PT, \qquad \operatorname{grade} PT < m$$

6. Another Generic Complex

Let F and G be free R-modules of ranks m and n, respectively, and let $\phi: F \to G$ be a map. Write $\phi_{r,s}$: $\wedge^r F \otimes \wedge^s G \to \wedge^{r+s} G$ for the "multiplication" map. It would be very desirable to know the minimal free resolution of coker $\phi_{r,s}$ under the assumption that its annihilator, which has the same radical as $I_{n-s}(\phi)$ by Proposition 1.5, has grade $(s+1)(m-n+s+1)$, the maximum possible. This was done in section 3 for $s = 0$ when $m \geqslant n$. (The case $s = 0$, $m \leqslant n$ can easily be deduced from this.) For $s = 1$, $r = n-1$, the desired complexes have been found for the cases when $m = n$ [14] or $m = n + 1$ [23].

In this section we will show how to construct free complexes $\mathbf{F}^p(\phi)$, $2 \leqslant p \leqslant n$, of length 4, which seem to be solutions to the problem for $s = 1$, $r = p - 1$, $m = n$. For $p = n$, $\mathbf{F}^n(\phi)$ will be the complex of

Gulliksen and Negård. For $p = 2$ we have also checked the grade sensitivity of $\mathbf{F}^p(\phi)$, so we have

PROPOSITION 6.1. *Suppose* $\phi: F \to G$ *is a map of free* R*-modules, and* $\operatorname{rank} F = \operatorname{rank} G = n$. *Suppose that* $\operatorname{grade}(I_{n-1}(\phi)) = 4$ *(as will be the case when* ϕ *is represented by a matrix of indeterminates). Then* $\mathbf{F}^n(\phi)$ *is a free resolution of* $R/I_{n-1}(\phi)$, *and* $\mathbf{F}^2(\phi)$ *is a free resolution of* $\wedge^2(\operatorname{coker}\phi)$.

If R is local with maximal ideal M, and if $\phi(F) \subset MG$, then $\mathbf{F}^p(\phi)$ will be a minimal complex for each p. Thus, in this case, the free resolutions of Proposition 6.1 will be minimal free resolutions. It would probably not be difficult to check the following.

Conjecture. Under the hypotheses of Proposition 6.1, $\mathbf{F}^p(\phi)$ is a free resolution of $\operatorname{coker}(\phi_{1,p-1})$ for each p between 2 and n.

We now turn to the details of the construction, which starts from one of the grade-sensitive complexes defined in [2] (which may well give nonminimal—in fact, infinite—resolutions of $\operatorname{coker}(\wedge^r F \otimes \wedge^s F \to \wedge^{r+s} G)$ for every r, s, m and n). We will begin with a description of the part of that complex which we need. For further details, the reader should consult [2].

We fix a map $\phi: F \to G$, where F and G are free modules of rank n, and we define for each $p \geqslant 2$ a complex:

$$\mathbf{C}^p: \cdots \to C_4{}^p \xrightarrow{\delta_3{}^p} C_3{}^p \xrightarrow{\delta_2{}^p} C_2{}^p \xrightarrow{\delta_1{}^p} C_1{}^p \xrightarrow{\delta_0{}^p} C_0{}^p$$

The terms $C_i{}^p$ and the maps $\delta_i{}^p$ are described as follows:

$$C_0{}^p = \bigwedge^{p} G$$

$$C_1{}^p = \bigwedge^{p} F \oplus \bigwedge^{p-1} F \otimes G$$

$$C_2{}^p = \bigwedge^{n-p+1} G^* \otimes \bigwedge^{n} F \otimes G \oplus \bigwedge^{p-1} F \otimes F$$

$$C_3{}^p = \bigwedge^{n-p+1} G^* \otimes \bigwedge^{n} F \otimes F \oplus \bigwedge^{p-1} F \otimes \bigwedge^{n} G^* \otimes \bigwedge^{n} F \otimes G$$

$$C_4{}^p = \bigwedge^{n-p+1} G^* \otimes \bigwedge^{n} F \otimes \bigwedge^{n} G^* \otimes \bigwedge^{n} F \otimes G \oplus \bigwedge^{p-1} F \otimes \bigwedge^{n} G^* \otimes \bigwedge^{n} F \otimes F$$

These are just the initial terms of the general complex described in [2] in the special case we are considering.

$\delta_0^p\colon C_1^p \to C_0^p$ is the map $\phi_{p,0} + \phi_{p-1,1}$

$\delta_1^p\colon C_2^p \to C_1^p$ is defined on generators by:

$\delta_1^p(\beta \otimes a \otimes b) = b(\beta)(a) + \beta(a) \otimes b$ and

$\delta_1^p(a_1 \otimes a_2) = a_1 \wedge a_2 + a_1 \otimes \phi(a_2)$

$\delta_2^p\colon C_3^p \to C_2^p$ is defined on generators by:

$\delta_2^p(\beta \otimes a_1 \otimes a_2) = \beta \otimes a_1 \otimes \phi(a_2) + \beta(a_1) \otimes a_2$ and

$\delta_2^p(a_1 \otimes \beta \otimes a_2 \otimes b) = a_1(\beta) \otimes a_2 \otimes b + a_1 \otimes b(\beta)(a_2)$

$\delta_3^p\colon C_4^p \to C_3^p$ is defined on generators by:

$\delta_3^p(\beta_1 \otimes a_1 \otimes \beta_2 \otimes a_2 \otimes b) = \beta_1 \otimes a_1 \otimes b(\beta_2)(a_2) + \beta_1(a_1) \otimes \beta_2 \otimes a_2 \otimes b$ and

$\delta_3^p(a_1 \otimes \beta \otimes a_2 \otimes a_3) = a_1(\beta) \otimes a_2 \otimes a_3 + a_1 \otimes \beta \otimes a_2 \otimes \phi(a_3)$

The terms C_i^p for $i > 4$ and the map δ_i^p for $i > 3$ are defined similarly (an explicit definition is given in [2]), but will not be needed for our purposes here.

Next we define a complex:

$$\mathbf{D}^p\colon \cdots \to D_4^p \xrightarrow{\partial_3^p} D_3^p \xrightarrow{\partial_2^p} D_2^p \xrightarrow{\partial_1^p} D_1^p \xrightarrow{\partial_0^p} D_0^p$$

The terms D_0^p and D_1^p are 0, and ∂_0^p and ∂_1^p are 0.

$$D_2^p = \bigwedge^{n-p} G^* \otimes \bigwedge^n F$$

$$D_3^p = \bigwedge^{p-1} F \otimes \bigwedge^{n-1} G^* \otimes \bigwedge^n F$$

To describe D_4^p, first set

$$X = \bigwedge^{n-q+1} G^* \otimes \bigwedge^n F \otimes \bigwedge^{n-1} G^* \otimes \bigwedge^n F \oplus \bigwedge^{p-1} F \otimes F \otimes \bigwedge^n G^* \otimes \bigwedge^n F$$

and let $Z(X)$ be the kernel of the map $\tau\colon X \to \wedge^p F \otimes \wedge^n G^* \otimes \wedge^n F$ defined by:

$$\tau(\beta_1 \otimes a_1 \otimes \beta_2 \otimes a_2) = \sum_{\deg \beta'_{1i} = n-p} \beta'_{1i}(a_1) \otimes \beta''_{1i} \otimes \beta_2 \otimes a_2$$

and

$$\tau(a_1 \otimes a_2 \otimes \beta \otimes a_3) = a_1 \wedge a_2 \otimes \beta \otimes a_3$$

The notation β'_{1i} and β''_{1i} means we are taking the terms in $\Delta(\beta_1) = \sum \beta'_{1i} \otimes \beta''_{1i}$ with degree $\beta'_{1i} = n - p$. We now set:

$$D_4{}^p = Z(X)$$

The map $\partial_2{}^p\colon D_3{}^p \to D_2{}^p$ is defined by

$$\partial_2{}^p(a_1 \otimes \beta \otimes a_2) = a_1(\beta) \otimes a_2$$

and $\partial_3{}^p\colon D_4{}^p \to D_3{}^p$ is defined by restriction to $Z(X)$ of the map, again denoted by $\partial_3{}^p$, of X into $D_3{}^p$ given on the generators by:

$$\partial_3{}^p(\beta_1 \otimes a_1 \otimes \beta_2 \otimes a_2) = \beta_1(a_1) \otimes \beta_2 \otimes a_2$$

and

$$\partial_3{}^p(a_1 \otimes a_2 \otimes \beta \otimes a_3) = a_1 \otimes a_2(\beta) \otimes a_3$$

Again, the terms $D_i{}^p$ for $i > 4$ are not needed here; we set them $= 0$ as well.

Finally, we define a trivial complex

$$\mathbf{E}^p\colon \cdots \to E_4{}^p \to E_3{}^p \to E_2{}^p \to E_1{}^p \to E_0{}^p$$

by setting $E_i{}^p = 0$ for $i \neq 1, 2$, $E_1{}^p = E_2{}^p = \wedge^p F$, and the map $E_2{}^p \to E_1{}^p$ is the identity.

With these complexes in hand, we define maps of complexes

$$\mathbf{D}^p \xrightarrow{v^p} \mathbf{C}^p \xrightarrow{u^p} \mathbf{E}^p$$

as follows:

$u_0{}^p = 0$; $u_1{}^p$ is the projection onto $\wedge^p F$, and $u_2{}^p = u_1{}^p\, \delta_1{}^p$. This describes u^p completely.

The maps $v_0{}^p$ and $v_1{}^p$ are, of course, zero. To define $v_2{}^p$, $v_3{}^p$ and $v_4{}^p$, we shall choose a basis $\{g_1, \ldots, g_n\}$ of G, and let $\{\gamma_1, \ldots, \gamma_n\}$ be the dual basis of G^*. (Although we shall define our maps $v_i{}^p$ using these bases, the reader will see that the maps really have an invariant definition.)

$$v_2{}^p\colon \bigwedge^{n-p} G^* \otimes \bigwedge^{n} F \to C_2{}^p \text{ is defined by}$$

$$v_2{}^p(\beta \otimes a) = \sum \beta \wedge \gamma_i \otimes a \otimes g_i + \sum_{\deg a_j' = n-1} \beta(a_j') \otimes a_j''$$

where again the notation a_j', a_j'' means we are taking the terms in $\Delta(a) = \sum a_j' \otimes a_j''$ with degree $a_j' = n - 1$.

$$v_3{}^p\colon \bigwedge^{p-1} F \otimes \bigwedge^{n-1} G^* \otimes \bigwedge^{n} F \to C_3{}^p \text{ is defined by}$$

$$v_3{}^p(a_1 \otimes \beta \otimes a_2) = \sum_{\deg a_{ij}' = p-2} a_{ij}'(\beta) \otimes a_2 \otimes a_{1j}'' + a_1 \otimes \sum_L \gamma_i \wedge \beta \otimes a_2 \otimes g_i$$

$v_4{}^p\colon Z(X) \to C_4{}^p$ is again going to be defined on X, and then restricted to $Z(X)$.

$$v_4{}^p(\beta_1 \otimes a_1 \otimes \beta_2 \otimes a_2) = \beta_1 \otimes a_1 \otimes \sum \gamma_i \wedge \beta_2 \otimes a_2 \otimes g_i$$

and

$$v_4{}^p(a_1 \otimes a_2 \otimes \beta \otimes a_3) = a_1 \otimes \beta \otimes a_3 \otimes a_2$$

With a little laborious calculation, one shows that the maps v^p and u^p are maps of complexes, and that $u^p v^p = 0$. We may therefore define a complex $\mathbf{F}^p(\phi)$ by setting:

$$\mathbf{F}^p(\phi)_i = \ker u_i{}^p / \operatorname{im} v_i{}^p \qquad \text{for} \quad i = 0, 1, 2, 3, 4$$

and

$$\mathbf{F}^p(\phi)_i = 0 \qquad \text{for} \quad i > 4$$

(If we had defined the terms $C_i{}^p$, $D_i{}^p$ and the maps $v_i{}^p$ for $i > 4$, it would have followed that $\mathbf{F}^p(\phi)_i = 0$ for $i > 4$.)

We then have:

$$\mathbf{F}^p(\phi)_0 = \bigwedge^p G$$

$$\mathbf{F}^p(\phi)_1 = \bigwedge^{p-1} F \otimes G$$

$$\mathbf{F}^p(\phi)_2 = L_2^{p-1}(F^*)^* \oplus A(G^*) \otimes \bigwedge^n F$$

where $A(G^*)$ is the cokernel of the map $\wedge^{n-p} G^* \to \wedge^{n-p+1} G^* \otimes G$ defined by $\beta \to \sum \gamma_i \wedge \beta \otimes g_i$. This is easily seen to be isomorphic to $L_2^{p-1}G$, and thus, $A(G^*)$ is a free module.

$$\mathbf{F}^p(\phi)_3 = \bigwedge^{n-p+1} G^* \otimes \bigwedge^n F \otimes F$$

$$\mathbf{F}^p(\phi)_4 = \bigwedge^p F \otimes \bigwedge^n G^* \otimes \bigwedge^n F$$

Thus, $\mathbf{F}^p(\phi)$ is a free complex of length 4 and agrees with the complex of [14] when $p = n$.

Remark. The term $D_4{}^p$ seems mysterious, but it is seen to be free for the same reason that $\ker u_2{}^p$ is seen to be free. [$\ker u_2{}^p$ is easily seen to be $L_2^{p-1}(F^*) \otimes \wedge^{n-p+1} G^* \otimes \wedge^n F \otimes G$.]

The proof of Proposition 6.1 is more calculation, using an idea that follows from [3] or from Lemma 4.1: Since the length of the complex $\mathbf{F}^p(\phi)$ is 4, and we are trying to prove it exact under the hypothesis that grade $I_{n-1}(\phi) = 4$, it is enough to show that $\mathbf{F}^p(\phi)$ is exact under the additional hypothesis that $I_{n-1}(\phi) = R$, or equivalently, it is enough to show that some power of $I_{n-1}(\phi)$ annihilates the homology of $\mathbf{F}^p(\phi)$. This may be done either by defining a homotopy on $\mathbf{F}^p(\phi)$ for multiplication by an element of $I_{n-1}(\phi)$, or by assuming that ϕ has, for a suitable choice of bases in F and G, the matrix:

$$\phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & & \\ 0 & 1 & & & 0 \\ & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & 0 & & 0 \end{pmatrix}$$

and proving the exactness of $\mathbf{F}^p(\phi)$ directly in this case.

A close examination of the complexes defined in [2] for a map $\phi: F \to G$ indicates that a resolution of $\operatorname{coker}(\wedge^{p-1} F \otimes G \to \wedge^p G)$, even when $m > n$, should always start out as:

$$\cdots \to L_2^{p-1}(F^*)^* \oplus A(G^*) \otimes \bigwedge^n F \to \bigwedge^{p-1} F \otimes G \otimes \bigwedge^p G \tag{6.1}$$

For $p = n$ and $m = n + 1$, these terms agree with those in the complex described in [23], despite the fact that they appear at first glance to be somewhat different.

It further appears, although our work in this direction is still very fragmentary, that in general a "resolution" of $\operatorname{coker}(\wedge^r F \otimes \wedge^s G \to \wedge^{r+s} G)$ should start out as:

$$\cdots \to \left[\bigwedge^{n-p+1} G^* \otimes \bigwedge^{n-s+1} F \otimes \bigwedge^s G \oplus \sum_{s-1 \geqslant t \geqslant 0} K_{p-t}^r F \otimes \bigwedge^t G\right] \Big/ T$$

$$\to \bigwedge^r F \otimes \bigwedge^s G \to \bigwedge^p G \tag{6.2}$$

where $p = r + s$, $K^r_{p-t}F = \ker(\wedge^r F \otimes \wedge^{p-r-t} F \to \wedge^{p-t} F)$ and T is a suitable submodule of the bracketed term such that the factor module is free. To make this last remark a bit clearer, we will show why the term $L_2^{p-1}(F^*) \oplus A(G^*) \otimes \wedge^n F$ in (6.1) is of the form indicated in (6.2).

Consider the module $\wedge^{n-p+1} G^* \otimes \wedge^n G \otimes G \oplus \wedge^{p-1} F \otimes F$. This maps to $\wedge^p F$ by sending $\beta \otimes a \otimes b \in \wedge^{n-p+1} G^* \otimes \wedge^n F \otimes G$ to $b(\beta)(a)$, and by sending $a_1 \otimes a_2 \in \wedge^{p-1} F \otimes F$ to $a_1 \wedge a_2$. We will call this map δ. The kernel of δ is isomorphic to $\wedge^{n-p+1} G^* \otimes \wedge^n F \otimes G \oplus K_p^{p-1}F$, and $K_p^{p-1}F$ is $L_2^{p-1}(F^*)^*$. The module $\wedge^{n-p} G^* \otimes \wedge^n F$ maps to $\wedge^{n-p+1} G^* \otimes \wedge^n F \otimes G \oplus \wedge^{p-1} F \otimes F$ by sending $\beta \otimes a$ to

$$\sum \beta \wedge \gamma_i \otimes a \otimes g_i + \sum_{\mathrm{d}\ \ \mathrm{g}\, a_i''=1} \beta(a_i') \otimes a_i'',$$

where $\{g_i\}$, $\{\gamma_i\}$ are dual bases of G and G^*, while the elements a_i' and a_i'' come from diagonalizing the element a. The image T, of this map is in $\ker \delta$, and $\ker \delta / T$ is $L_2^{p-1}(F^*)^* \oplus A(G^*) \otimes \wedge^n F$.

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