A Survey of Some Results on Free Resolutions*

David Eisenbud**

Consider the task of solving a system of linear homogeneous equations of rank r in n variables over a field K or, equivalently, of finding the kernel of a homomorphism of rank r of vector spaces: $K^n \to K^m$. As everyone knows, three facts make this process simple:

(0) There are $n - r$ linearly independent solutions which span the space of solutions.

(1) It is easy to tell whether a given set of solutions spans all the solutions: It does if and only if it spans a space of dimension $n - r$. Equivalently, if $\psi : K^p \to K^n$ is a map such that $\varphi\psi = 0$, then

$$K^p \xrightarrow{\psi} K^n \xrightarrow{\varphi_0} K^m$$

is exact if and only if rank $\psi = n - r$.

(2) There is a "formula," in terms of the minors of φ, for a map ψ making the above sequence exact: For example, ψ can be taken as the map

$$\wedge^r K^{m*} \otimes \wedge^{r+1} K^n \xrightarrow{\psi_0} K^n$$

induced by φ. (Here * denotes $\mathrm{Hom}_K(-, K)$.)

In this note we will sketch some results from the theory of finite free resolutions which are analogues of (0), (1), (2) for rings more general than fields. We will also outline a technique for dealing with finite free resolutions that does not seem to have an interesting vector space analogue, and exhibit one of the interesting phenomena that arise when one works with infinite instead of with finite resolutions.

Throughout this paper, R will be a local Noetherian ring (the restrictions could

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be largely relaxed) and all modules will be finitely generated R-modules.

I. Some analogues. A map $F_1 \xrightarrow{\varphi_1} F_0$ of free R-modules (which may be considered as a system of homogeneous linear equations over R) will not in general have any of the nice properties (0), (1), (2). Because (0) may fail, a study of the solutions to the equations φ_1 must include a study of the dependence relations between the elements of a minimal set of solutions which spans all the solutions; that is, "solving" the equations must be taken to mean finding a whole free resolution

$$\boldsymbol{F}: \cdots F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

of coker φ_1. Once this has been said, there is an analogue of (0) holding for "nice" local rings.

THEOREM (AUSLANDER-BUCHSBAUM-SERRE). *If R is a regular local ring of dimension d, and $\boldsymbol{F}$ is a free resolution as above, with* rank $\varphi_i = r_i$, *then for all $i \geqq d - 1$,* ker φ_i *is spanned by* (rank F_i) $- r_i$ *linearly independent solutions; in other words* ker φ_i *is free.* (*Here as always the* rank *of φ_i is the largest integer r with $\wedge^r \varphi_i \neq 0$.*)

Since "solving a system of equations" means finding a free resolution, it is clear that the analogue of property (1) of the introduction should be a criterion for exactness of a complex of free modules:

$$\boldsymbol{G}: \cdots \xrightarrow{\varphi_3} G_2 \xrightarrow{\varphi_2} G_1 \xrightarrow{\varphi_1} G_0.$$

In general, no such criterion is known, but there is such a criterion if $\boldsymbol{G}$ is finite—that is, if $G_i = 0$ for $i \gg 0$. We need one more definition before we can state it: If $\varphi: F \to G$ is a map of free modules with rank $\varphi = r$, we define $I(\varphi)$ to be the ideal generated by the $r \times r$ minors of φ. Intrinsically put, $I(\varphi)$ is the image of the map

$$\wedge^r G^* \otimes \wedge^r F \to \wedge^0 G = R$$

induced by φ.

To state the theorem we must also recall that the *grade* of a proper ideal I of R is by definition the length of a maximal R-sequence in I. If $I = R$, we make the convention grade $I = \infty$. Or, simply define

$$\text{grade } I = \inf\{g \mid \text{Ext}^g(R/I, R) \neq 0\}.$$

THEOREM [B-E 1]. *With the above notations, suppose that $G_i = 0$ for $i \gg 0$. Then $\boldsymbol{G}$ is exact if and only if for each $k \geqq 1$*

$$(1) \qquad\qquad \text{rank } \varphi_{k+1} + \text{rank } \varphi_k = \text{rank } G_k,$$
$$(2) \qquad\qquad \text{grade } I(\varphi_k) \geqq k.$$

The situation with regard to an analogue of property (2) of the introduction is both more complicated and less satisfactory. Given a map $\varphi: F \to G$ of rank r, we may still construct the map $\psi_0: \wedge^r G^* \otimes \wedge^{r+1} F \to F$, and we will still have $\varphi \psi_0 = 0$. It is easy to see, however, that

$$\wedge^r G^* \otimes \wedge^{r+1} F \xrightarrow{\psi_0} F \xrightarrow{\varphi} G$$

need not be exact. (If rank $G = r$, this sequence is exact if and only if grade $I(\varphi) =$ (rank $F) - r + 1$, the largest possible value; see [B-R].) It is even easy to give examples of maps φ_i such that if $F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ is exact, then the ideal generated by the entries of a matrix for φ_2 is not contained in the ideal generated by the entries of a matrix for φ, so that φ_2 cannot be derived from φ_1 by a "formula" in any ordinary sense. To get an idea of what sort of thing might be true about the relation of φ_1 to φ_2, consider the following very useful theorem, which was proved in a special case by Hilbert, and extended to the general case by Burch [Bur]. This theorem has been a model for much of the work on finite free resolutions; it gives a sort of parametrization of ideals of projective dimension 1 which has been applied, for instance, to the study of deformations, residual complete intersections, factoriality, and the Zariski-Lipman conjecture.

THEOREM (HILBERT-BURCH). *Let*

$$F: 0 \to R^{n-1} \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi_1} R$$

be a complex. F *is exact if and only if* F *is isomorphic to a complex of the form*

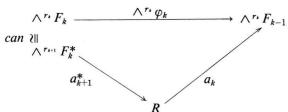$$0 \to R^{n-1} \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi_1} R$$

where φ_1' *is the composite map*

$$R^n \cong \wedge^{n-1} R^{n*} \xrightarrow{\wedge^{n-1}\varphi_2^*} \wedge^{n-1} R^{n-1*} \xrightarrow{can} R \xrightarrow{a} R,$$

with φ_1' denoting the overall composite,

where a *is a non-zero-divisor,* rank $\varphi_2 = n - 1$, *and* grade $I(\varphi_2) = 2$.

The essential point of this theorem is that if F is exact, there is a factorization of φ_1 through $\wedge^{n-1} \varphi_2^*$. Noting that rank $\varphi_1 = 1$, we see that the following theorem extends this result to a result for all finite free resolutions:

THEOREM ([B-E 2], [E-N]). *Let* $F: 0 \to F_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_1} F_0$ *be a finite free resolution, and set* $r_k = $ rank φ_k. *Then there are unique maps* $a_k: R \to \wedge^{r_k} F_{k-1}$ *such that the diagrams*

$$
\begin{array}{ccc}
\wedge^{r_k} F_k & \xrightarrow{\wedge^{r_k}\varphi_k} & \wedge^{r_k} F_{k-1} \\
can \,\, \wr\| & & \\
\wedge^{r_{k+1}} F_k^* & & \\
& a_{k+1}^* \searrow \quad \nearrow a_k & \\
& R &
\end{array}
$$

commute. (The canonical isomorphisms exist because $r_k + r_{k+1} = $ rank F_k.) *Moreover,* a_n *is the composite*

$$R \cong \wedge^{r_n} F_n \xrightarrow{\wedge^{r_n}\varphi_n} \wedge^{r_n} F_{n-1},$$

and

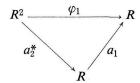$$\operatorname{Rad} I(a_k) = \operatorname{Rad} I(\varphi_k).$$

This extension of the Hilbert-Burch theorem is useful in several contexts; for example, it gives another proof of MacRae's strengthening of the theorem that a regular local ring is a unique factorization domain:

COROLLARY. *Let R be a local ring, I an ideal generated by 2 elements. If p.d. $R/I <$ ∞, then p.d. $R/I \leqq 2$, and I is isomorphic to an ideal J generated by an R-sequence.*

PROOF [**B-E 2**]. Apply Theorem 2 to a finite free resolution

$$\cdots \to R^n \xrightarrow{\varphi_2} R^2 \xrightarrow{\varphi_1} R \to R/I \to 0,$$

obtaining a factorization

$$\begin{array}{ccc}
R^2 & \xrightarrow{\ \varphi_1\ } & R \\
{\scriptstyle a_2^*}\searrow & & \nearrow{\scriptstyle a_1} \\
 & R &
\end{array}$$

since a_2 also enters into a factorization of $\wedge^{r_2}\varphi_2$, $I(a_2^*) = J$ will be an ideal generated by an R-sequence x, y of length 2, and R/I will have a resolution of the form

$$0 \to R \xrightarrow{a_2} R^2 \xrightarrow{\varphi_1} R \to R/I \to 0. \quad \square$$

This theorem has recently been applied by Hochster [**Hoc**] to the construction of the "generic" free resolutions of length 2, which indicates that in a sense this theorem is a "complete" result for resolutions of length 2. However, for longer resolutions, or to achieve a direct generalization of the Hilbert-Burch theorem for ideals I with p.d. $I > 1$, more is needed. Since the "right" theorem of this type has not been found as yet, we content ourselves with an illustration of what may be done. See [**B-E 2**] for more results of this type.

THEOREM. *Let $0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} F_{n-2} \to \cdots \to F_0$ be a free resolution, and set $r_k = \operatorname{rank} \varphi_k$. Then the map $\wedge^{r_{n-1}-i}\varphi_{n-1}$ can be factored through the dual of the map*

$$\wedge^i F_{n-1} \otimes \wedge^{r_n} F_n \xrightarrow{\ 1 \otimes \wedge^{r_n}\varphi_n\ } \wedge^i F_{n-1} \otimes \wedge^{r_n} F_{n-1},$$

for $i \leqq n - 2$.

II. Some new directions.

(A) *Algebra structures on free resolutions.* Let $\boldsymbol{F}: \cdots \to F_1 \to R$ be a free resolution of a cyclic module. The symmetric square $S_2(\boldsymbol{F})$ is a complex of free R-modules, and it agrees with $\boldsymbol{F}$ in degrees 0 and 1. There is thus a comparison map, unique up to homotopy, $S_2(\boldsymbol{F}) \xrightarrow{\nu} \boldsymbol{F}$, which may be regarded as equipping $\boldsymbol{F}$ with the structure of a (strictly skew-) commutative, homotopy-associative, differential graded algebra. We do not know whether ν may be chosen to make $\boldsymbol{F}$ associative as well, except in a few cases such as that of the minimal free resolution of the residue class field of R.

Using this idea, it is possible to obtain results like the Hilbert-Burch theorem for certain classes of resolutions of length 3. For example, suppose that

$$F: 0 \to R \to R^n \xrightarrow{\varphi_1} R^n \xrightarrow{\varphi_2} R$$
$$\parallel \quad \parallel \quad \parallel \quad \parallel$$
$$F_3 \quad F_2 \quad F_1 \quad F_0$$

is a minimal free resolution such that the complex $F^* = \text{Hom}(F, R)$ is exact. (If R is a regular ring, and $\varphi_1: R^n \to R$, then coker φ_1 will have a minimal free resolution of this form if and only if coker φ_1 is a Gorenstein factor ring of R of codimension 3.) The algebra structure on F induces isomorphisms $F_i \to F_{3-i}^*$, and the commutativity of this algebra structure ensures that the composite map

$$\overbrace{F_1^* \to F_2 \cong F_1}^{\varphi}$$

is alternating. It turns out that n is odd and that im φ_1 is generated by the "principal $n - 1 \times n - 1$ pfaffians" of φ, from which one obtains a theorem parallel to the Hilbert-Burch theorem [B-E 3].

It seems possible that this "multiplicative" technique, together, perhaps, with the idea of Liason of Peskine-Szpiro [P-S], may eventually give structure theorems along the lines of the Hilbert-Burch theorem for all free resolutions of the form $F: 0 \to F_3 \to F_2 \to F_1 \to R$ with F^* exact—that is, for all perfect ideals of grade 3. The first step in this program beyond the Gorenstein case is worked out in [B-E 3].

(B) *A glimpse of the infinite case.* The only infinite free resolutions that have received much attention so far are the free resolutions of the residue class fields of (nonregular) local rings. As already remarked, these have the structure of associative algebras, and in fact they are free algebras, on generators of various degrees, about which much may be said (see [G-L] for an exposition). For our purposes we single out the following result:

THEOREM (BRAUER, GULLIKSEN). *Let R be a local ring, and let $F: \cdots \to F_1 \to R \to k \to 0$ be a minimal free resolution of the residue class field k of R. Then the numbers $b_k = \text{rank } F$ are bounded if and only if the maximal ideal of R is generated by at most $1 + \dim R$ elements. If this is so then $b_k = b$, a constant, for all $k \geq 1 + \dim R$, and there are isomorphisms*

$$F_k \xrightarrow{\alpha_k} F_{k+2} \qquad (k \geq 1 + \dim R)$$

such that

$$\varphi_{k+2} = \alpha_{k-1} \varphi_k \alpha_k^{-1} \qquad (k > 1 + \dim R).$$

In fact, something similar holds not only for the residue class field, but also for every module, and a little more is true. For the purpose of this theory we may harmlessly pass to the completion of R, after which, assuming that the maximal ideal of R can be generated by $1 + \dim R$ elements, we may write $R = S/(s)$, where S is a regular local ring and $s \in S$. We now have

THEOREM. *Let S be a regular local ring of dimension d, and let $s \in S$ be an element.*

Set $R = S/(s)$. Let $\boldsymbol{F}: \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$ be a minimal free resolution over R. Then rank $F_k =$ rank F_{k+1} for all $k \geq d$, and there exist isomorphisms $\alpha_k : F_k \longrightarrow F_{k+2}$ ($k \geq d$) such that $\varphi_{k+2} = \alpha_{k-1} \varphi_k \alpha_k^{-1}$ ($k > d$). Moreover there are liftings of the maps

$$F_{d+2} \xrightarrow{\varphi_{d+2}} F_{d+1} \xrightarrow{\alpha_d \varphi_{d+1}} F_{d+2}$$

to maps of free S-modules

$$\tilde{F}_{d+2} \xrightarrow{\tilde{\varphi}_{d+2}} \tilde{F}_{d+1} \xrightarrow{\tilde{\varphi}_{d+1}} \tilde{F}_{d+2}$$

such that $\tilde{\varphi}_{d+1} \tilde{\varphi}_{d+2} = s \cdot I = \tilde{\varphi}_{d+2} \tilde{\varphi}_{d+1}$, where I is the identity map.

This theorem may in fact be pushed a little farther to yield a one-to-one correspondence between Cohen-Macaulay R-modules of dimension $d - 1$, and factorizations in the matrix ring over S of the scalar matrix $s \cdot I$.

It seems reasonable to conjecture that some result of this type should hold for rings R which can be written as a quotient of a regular local ring S by an S-sequence: If

$$\boldsymbol{F}: \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

is a free resolution over such a ring R, then for $k \gg 0$, there should be a formula for φ_k in terms of $\varphi_1, \cdots, \varphi_{k-1}$ as *is* the case for resolutions of the residue class fields.

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BRANDEIS UNIVERSITY

WALTHAM, MASSACHUSETTS 02154, U.S.A.