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Some Structure Theorems for Finite Free Resolutions

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CONTENTS

INTRODUCTION

In this paper we investigate the relationships that hold between the homomorphisms in an exact sequence of the form:

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0, \tag{0.1}$$

where each F_k is a finitely generated free module over a commutative noetherian ring R. Such a sequence is a *finite free resolution* of the cokernel of f_1.

The starting point of this article was the following beautiful result of Hilbert [18].

THEOREM 0 (Hilbert). *Let R be a commutative noetherian ring. If*

$$0 \longrightarrow R^n \xrightarrow{f_2} R^{n+1} \xrightarrow{f_1} R \longrightarrow R/I \longrightarrow 0 \tag{0.2}$$

is a free resolution of a cyclic module R/I, where I is an ideal of R, then I is *a multiple of the ideal generated by the $n \times n$ minors of the matrix f_2; more precisely, there exists a nonzero-divisor $a \in R$ such that the image of the* i-th *basis element of R^{n+1}, under the map f_1, is $a\Delta_i$, where Δ_i is the $n \times n$ minor of the matrix f_2 obtained by leaving out the* i-th *row.*

Modern treatments of Theorem 0 have not been lacking. Reference [11] seems to contain the first proof in the generality in which we have stated the result. The fastest proof is outlined in [19, Exercise 8, p. 148].

A number of interesting applications of Theorem 0 are given in the literature. For example, it has been used in an attack on Grothendieck's lifting problem [6, 7] and in the study of deformations of space curves [23]. Unfortunately, Theorem 0 gives no insight into ideals of homological dimension greater than 1, so applications like those above have been limited to low-dimensional situations.

In this paper we use a combination of old linear algebra and modern homological ring theory to obtain results which extend Theorem 0 to deal with arbitrary finite free resolutions. From these, we deduce several consequences for the structure of ideals.

We will now describe our results. Theorem 0 asserts the existence of a nonzero divisor $a \in R$ which gives rise to a factorization of the map f_1 in (0.2) of the form

$$\begin{tikzcd} R^{n+1} \arrow[rr, "f_1"] \arrow[d, equal] & & R \\ \bigwedge^n R^{n+1*} \arrow[r, "\bigwedge^n f_2^*"'] & R \arrow[ur, "a"'] & \end{tikzcd}$$

Our generalization of Theorem 0 asserts the existence, for an arbitrary finite free resolution (0.1), of a sequence of unique maps, a_k, which give rise to factorizations:

$$\begin{tikzcd} \dots, \bigwedge^{r_k} F_k \arrow[rr, "\bigwedge^{r_k} f_k"] \arrow[dr, "a_{k+1}^*"'] & & \bigwedge^{r_k} F_{k-1}, \\ & R \arrow[ur, "a_k"'] & \end{tikzcd} \qquad \begin{tikzcd} \bigwedge^{r_{k-1}} F_{k-1} \arrow[rr, "\bigwedge^{r_{k-1}} f_{k-1}"] \arrow[dr, "a_k^*"'] & & \bigwedge^{r_{k-1}} F_{k-2}, \dots, \\ & R \arrow[ur, "a_{k-1}"'] & \end{tikzcd}$$

where r_k is the rank of f_k, $\wedge^r F$ is the r-th exterior power of F, and $*$ denotes $\hom_R(-, R)$. (We have identified $\wedge^{r_{k-1}} F_{k-1}$ with $\wedge^{r_k} F_{k-1}^*$, using the fact, which follows from Theorem 1.2, that $r_k + r_{k-1} = \operatorname{rank} F_{k-1}$.)

In the special case dealt with by Theorem 0, $r_1 = 1$, so that $\wedge^{r_1} f_1 = f_1$ is the map which is factored.

If one chooses bases for the free modules F_k, then the entries of the matrices of the maps $\wedge^{r_k} f_k$, which are factored above, are the $r_k \times r_k$ minors of f_k. One reason for the appearance of the $r_k \times r_k$ minors of f_k in Theorem 3.1 is that the ideal generated by these minors—which we will call $I(f_k)$—plays a large rôle in determining the exactness of (0.1). More precisely, the theorem of [8], which we recapitulate here as Theorem 1.2, shows, for example, that if the ring R is an integral domain, then (0.1) is exact if and only if it is exact over the quotient field of R and the ideals $I(f_k)$ are "sufficiently large."

Theorem 1.2 is fundamental to the rest of the paper; all our main results turn out to depend for their proofs on the statement that something is exact, and we always use Theorem 1.2 or its corollary to verify this.

Section 2 contains a further result on the ideals $I(f_k)$, which does not require the finiteness of the resolution (0.1).

In Section 3 we present our first structure theorem, stated above, for finite free resolutions. The first part of the section is concerned with the relation between Theorem 3.1 and a well-known result on Plücker coordinates. Theorem 3.1 is stated and proved in the remainder of the section, modulo a certain lemma from linear algebra. Section 4 deals with this Lemma, which has further applications in Sections 6, 7, and 10.

The fifth section of the paper is concerned with the consequences of Theorem 3.1. Among these are a new proof of a theorem, due to MacRae [21], that every ideal with a finite free resolution has a "greatest common divisor." This yields, in particular, a new proof that a regular local ring is factorial. We also illustrate the usefulness of Theorem 3.1 in computation by showing that the cokernel of a generic $n \times m$ matrix of rank k, with $0 < k < \min(n, m)$ has infinite homological dimension.

Theorem 3.1 does not give a structure theorem for ideals of large homological dimension in the same sense that Theorem 0 gives a structure theorem for ideals of homological dimension one. The difficulty stems from the fact that Theorem 3.1 does not give information about the minors of order one of f_k, but about those of order r_k. In Section 6 we present a second extension of Theorem 0, which shows the existence of certain factorizations, for $k \geqslant 2$, of $\wedge^{r_k - 1} f_k$ (Theorem 6.1).

We exploit these results on minors of lower order in Sections 7, 8, and 9. In particular, Theorem 7.1 contains information on ideals with 3 generators which leads to a fairly complete structure theorem in homological dimension 2. As a result, we are able, in Section 9, to solve Grothendieck's lifting problem for 3-generator ideals of homological dimension 2.

In Section 10, we sketch a result on lower order minors which can be obtained by methods analogous to those used in the proofs of Theorems 3.1 and 6.1. The result includes Theorems 3.1 and 6.1 and is, in a sense, the best possible. But the information it gives is inadequate for the purposes of structure theory, and some new approach seems to be required for further progress.

Finally, in Section 11, we give a number of examples of finite free resolutions, and discuss some of the many problems that they suggest.

Another possible approach to the study of finite free resolutions is the study of the algebra structures that they support. We have pursued this point of view in [10], obtaining, in particular, a complete structure theorem for Gorenstein ideals of height 3 in a regular local ring.

Although we have restricted ourselves to the consideration of free resolutions, the interested reader will have no difficulty in modifying our theorems so that they hold for arbitrary finite projective resolutions. See the remarks at the ends of Sections 3 and 6 for more discussion of this point.

Good general references for the theory of exterior algebras, which we will use heavily, are [BOU 4, Sects. 7–11; and 6, Sect. 1].

The reader may wish to consult [14] for an exposition, based on a different and in some ways more elementary point of view, of some of the results of this paper.

1. How to Prove that a Complex is Exact

In this section we introduce the technique that we will use in the rest of this paper for proving that a finite complex of projective modules is exact.

We begin with some notation and terminology. Let R be a commutative ring. For any R-module M, we set $M^* = \hom_R(M, R)$. If P is a projective R-module, we will say that the rank of P is r if $\wedge^{r+1} P = 0$ but $\wedge^r P \neq 0$. We say that P has well-defined rank if, for every prime ideal x of R, the rank of P_x is the same as the rank of P.

If $f: P \to Q$ is a homomorphism of R-modules, then for each positive integer k, the map $\wedge^k f: \wedge^k P \to \wedge^k Q$ induces a map $(\wedge^k Q)^* \otimes \wedge^k P \to R$ whose image we call $I_k(f)$. If P and Q are free modules of finite rank, then choosing bases for P and Q, we may identify f with a matrix, and $I_k(f)$ is nothing but the ideal of $k \times k$ minors of this matrix. It is clear, in this case, that the ideals $I_k(f)$ are fitting invariants of Coker f, and depend, therefore, only on Cok f.

Returning to a map $f: P \to Q$ of R-modules we define the rank of f to be k if $\wedge^{k+1} f = 0$ but $\wedge^k f \neq 0$. Such a k must exist if P or Q is finitely generated. We set $I(f) = I_r(f)$ where $r = \operatorname{rank}(f)$. Note that if f is the zero map, then rank $f = 0$ and $I(f) = R$. The significance of the invariant $I(f)$ comes from the following well-known lemma.

LEMMA 1.1 [8]. *Let R be a commutative ring with no nontrivial idempotents and suppose $f: P \to Q$ is a homomorphism of finitely generated projective R-modules. Then* Cok f *is projective if and only if $I(f) = R$.*

It follows easily that if Cok f has well-defined rank, the condition that R have no nontrivial idempotent may be dropped.

We also note that if $\operatorname{rank}(f) = \operatorname{rank} Q$, then $I(f)$ has the same radical as $\operatorname{ann}_R(\operatorname{Cok} f)$.

Now suppose that R is a noetherian ring, and I a proper ideal of R. We say the depth of I is d if I contains an R-sequence of length d but no R-sequence of length $d + 1$. It is easily shown [19, Appendix 3-1], that I has depth d if and only if d is the smallest integer such that $\operatorname{Ext}_R{}^d(R/I, R) \neq 0$. Accordingly we make the convention that the depth of R, considered as an ideal of itself, is infinite.

We will now review some of the linear algebra that we will use. An *oriented free module* is by definition a free module P with a given choice of a generator $\eta \in \wedge^r P$, where $r = \operatorname{rank} P$; η is the orientation of P.

Now suppose that P is an oriented free module of rank r. The orientation of P determines, for each integer k between 0 and r, an isomorphism:

$$\bigwedge^k P \cong \bigwedge^{r-k} P^*.$$

Using this isomorphism, we will always identify $\wedge^k P$ with $\wedge^{r-k} P^*$.

An explicit description of this identification, as well as other basic facts of linear algebra, will be given in Section 4.

We are ready to state our main criterion for exactness:

THEOREM 1.2 [8]. *Let R be a noetherian ring and let*

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \tag{1.1}$$

be a complex of finitely generated free R-modules and nonzero maps. The sequence (1.1) *is exact if and only if for $k \geqslant 1$:*

(1) $\operatorname{rank}(f_k) + \operatorname{rank}(f_{k+1}) = \operatorname{rank} P_k$,

(2) $\operatorname{depth} I(f_k) \geqslant k$.

Remarks. (0) In [8] Theorem 1.2 was proved for a finite complex of finitely generated projective modules of well-defined rank.

(1) A similar theorem characterizes the exactness of a complex of the form (1.1) $\otimes M$, where M is any finitely generated R-module. See [8] for details.

(2) We have been informed by M. Hochster that a theorem similar to Theorem 1.2 holds for nonnoetherian rings provided that the right definition of depth is chosen.

(3) The depth of an ideal of the form $I(f)$, where $f: P \to Q$ is a map between projective modules of ranks p and q, cannot be arbitrary. In fact, it is known [13] that for such an f,

$$\operatorname{depth} I_k(f) \leqslant (p - k + 1)(q - k + 1).$$

We will often apply Theorem 1.2 in the following less concrete version.

COROLLARY 1.3. *Let* (1.1) *be a complex as in Theorem* 1.2. *Then* (1.1) *is exact if and only if for all primes x of R with depth $(xR_x) < n$, the localized complex $R_x \otimes$* (1.1) *is exact.*

Proof. To show that (1.1) is exact we show that the conditions of Theorem 1.2 are satisfied. If $\operatorname{depth} I(f_k) < k$ for some k, with $1 \leqslant k \leqslant n$, then there is a prime ideal x of R such that $x \supset I(f_k)$, and x is associated to a maximal R-sequence contained in $I(f_k)$. Hence $\operatorname{rank}((f_k)_x) = \operatorname{rank}(f_k)$ and depth $xR_x = \operatorname{depth} I(f_k) < k$. Thus

$$I((f_k)_x) = (I(f_k))_x \subset xR_x,$$

and depth $I((f_k)_x) < k$, contradicting Theorem 1.2 and the assumption that (1.1) $\otimes R_x$ is exact. Therefore, conditions (1) and (2) are satisfied by (1.1) and we have Corollary 1.3.

Remark. Corollary 1.3 could also be deduced from the "Lemme d'Acyclicité" of Peskine–Szpiro [22], which says that if R is a local ring whose maximal ideal has depth $\geqslant n$, and if $H_i((1.1))$ has depth 0 for all i, then (1.1) is exact.

2. Remarks on the Ideals $I(f_k)$

In this section we will prove a result on the ideals $I(f_k)$ associated to the maps f_k which holds in quite a general setting.

THEOREM 2.1. *Let R be a commutative ring without proper idempotents, and let*

$$P: \quad \cdots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0$$

be an exact sequence of finitely generated free R-modules. Suppose that for all $k > 1$, $I(f_k)$ contains a nonzero-divisor. Then

(a) *For all $k > 1$,* $\operatorname{Rad}(I(f_k)) \subseteqq \operatorname{Rad}(I(f_{k+1}))$.

(b) *Suppose in addition that R is noetherian and that rank $f_1 =$ rank P_0. If depth $(I(f_1)) = k$ then*

$$\operatorname{Rad}(I(f_1)) = \operatorname{Rad}(I(f_2)) = \cdots = \operatorname{Rad}(I(f_k)).$$

Remark. By Theorem 1.2, the hypothesis that each $I(f_k)$ contains a nonzero divisor is automatically satisfied in the case of most interest, when R is noetherian and P has only finitely many nonzero terms.

Proof of Theorem 2.1. (a) Suppose that $w \in \operatorname{Rad}(I(f_k))$, and let ω be the multiplicatively closed subset of R generated by w. We have $I((f_k)_\omega) = (I(f_k))_\omega = R_\omega$, so by Lemma 1.1, $\operatorname{Cok}((f_k)_\omega)$ is projective over R_ω. Since P_ω is exact, it follows that $\operatorname{Cok}((f_{k+1})_\omega)$ is also projective, so that by Lemma 1.1, $I((f_{k+1})_\omega) = R_\omega$. Since $I(f_{k+1})$ contains a nonzero divisor, we have rank $f_{k+1} = \operatorname{rank}((f_{k+1})_\omega)$. This implies that

$$(I(f_{k+1}))_\omega = I((f_{k+1})_\omega),$$

so $(I(f_{k+1}))_\omega = R_\omega$. It follows that $w \in \operatorname{Rad}(I(f_{k+1}))$.

(b) We will make use of two well-known facts: If rank $f_1 =$ rank P_0, then $I(f_1)$ annihilates $\operatorname{Cok}(f_1)$. Further, if a nonzero, finitely generated module M, over a noetherian ring R, is annihilated by an ideal of depth k, then $hd_R(M) \geqslant k$.

By virtue of part (a), it suffices to show that $\mathrm{Rad}(I(f_1)) \supseteq \mathrm{Rad}(I(f_k))$. Suppose that this were not so, and let $w \in \mathrm{Rad}(I(f_k))$, but $w \notin \mathrm{Rad}(I(f_1))$. By localizing at the multiplicatively closed set generated by w, we may suppose that $I(f_k) = R$, but that $I(f_1)$ is a proper ideal. Since $I(f_1)$ is proper, $\mathrm{Cok}\, f_1 \neq 0$, and since depth $I(f_1) = k$ by hypothesis, we have $hd(\mathrm{Cok}(f_1)) \geqslant k$. On the other hand, since $I(f_k) = R$, it follows from Lemma 1.1 that $\mathrm{Cok}\, f_k$ is projective; that is, $hd(\mathrm{Cok}(f_1)) < k$, a contradiction.

3. The First Structure Theorem

In this section we prove our first theorem describing the arithmetic structure of free resolutions. The theorem may be regarded as asserting that the solutions to certain equations that exist in the ring of quotients of a noetherian ring R actually exist in R itself.

We begin with an illustration. Let K be a field and let $V\colon 0 \to V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0$ be an exact sequence of finite-dimensional K-vector spaces. The subspace $\mathrm{Im}(f_2)$ is determined by its Plücker coordinates, which are obtained as follows: Choose bases for V_2 and V_1, and identify f_2 with its matrix with respect to these bases. The *Plücker coordinates* of $\mathrm{Im}(f_2)$ are the $p \times p$ minors of this $n \times p$ matrix, where $p = \dim V_2$ and $n = \dim V_1$.

On the other hand, choosing bases for V_1 and V_0, we may identify f_1 with a $q \times n$ matrix, where $q = \dim V_0$. The *dual Plücker coordinates* of $\ker f_1$ are the $q \times q$ minors of the matrix for f_1.

Since $\ker f_1 = \mathrm{Im}\, f_2$, the Plücker coordinates determine the dual Plücker coordinates up to a common multiple.

To make this more explicit, we introduce some notation. For any subset I of p elements of the chosen basis of V_1, let I' be the set of q remaining elements. We will write $[f_2, I]$ for the $p \times p$ minor of f_2 involving the rows of the matrix of f_2 that correspond to elements of I. Similarly, we write $[f_1, I']$ for the $q \times q$ minor of f_1 involving the columns of the matrix of f_1 which correspond to the elements of I'.

The relationship between the Plücker coordinates of $\mathrm{Im}(f_1)$ and the dual Plücker coordinates of $\ker(f_2)$ is easy to state: there exists an element $a \in K$ such that for each subset I of p elements of the basis of V_1,

$$[f_1, I'] = \pm a[f_2, I].$$

Now suppose that K is the quotient field of a noetherian ring R, and that

$$F\colon\ 0 \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

is an exact sequence of free R-modules such that

$$F_i \otimes K \cong V_i \qquad \text{and} \qquad \operatorname{rank}(f_1) = \operatorname{rank} F_0\,.$$

Choosing R-bases of F_2, F_1, F_0, we may identify f_2 and f_1 with $n \times p$ and $q \times n$ matrices of elements of R, so that if I is a set of elements of the basis of F_1, the minors $[f_2, I]$ and $[f_1, I']$ are elements of R. By what has gone before, there exists an element $a \in K$ such that $[f_1, I'] = \pm a[f_2, I]$. In this situation, Theorem 3.1 will say that a is an element of R.

One further remark is necessary before we state the theorem: if $P_2 \to^{f_2} P_1 \to^{f_1} P_0$ is part of a finite free resolution, and if $r_i = \operatorname{rank}(f_i)$ then by Theorem 1.2, $r_1 + r_2 = \operatorname{rank} P_1$. Thus, if P_1 is oriented we have the canonical identification:

$$\bigwedge^{r_1} P_1 \approx \bigwedge^{r_2} P_1{}^*.$$

THEOREM 3.1. *Let R be a commutative, noetherian ring, and let*

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \tag{3.1}$$

be an exact sequence of oriented free R-modules. Then:

(a) *for each k, $1 \leqslant k < n$, there exists a unique homomorphism a_k: $R \to \wedge^{r_k} P_{k-1} = \wedge^{r_{k-1}} P_{k-1}^*$ such that*

(i) $a_n = \wedge^{r_n} f_n : R = \wedge^{r_n} P_n \to \wedge^{r_n} P_{n-1}$

(ii) *for each $k < n$, the diagram*

$$\begin{tikzcd} \bigwedge^{r_k} P_k \arrow[rr, "\wedge^{r_k} f_k"] \arrow[dr, "a_{k+1}^*"'] & & \bigwedge^{r_k} P_{k-1} \\ & R \arrow[ur, "a_k"'] & \end{tikzcd}$$

commutes.

(b) *For all $k > 1$,* $\operatorname{Rad}(I(a_k)) = \operatorname{Rad}(I(f_k))$.

We will give a construction of a_{k-1} from a_k; thus starting with the definition of a_n given by (i), we will construct all the a_k. In order to make the idea behind the construction clear, we begin by indicating how to obtain a_{n-1} from a_n.

First, the map $f_n{}^*\colon P^*_{n-1} \to P_n{}^*$ gives rise to a complex ([14]; see [9] for an exposition in the spirit of this paper) the first few terms of which are

$$\cdots \to P_n \otimes \bigwedge^{r_n+1} P^*_{n-1} \to \bigwedge^{r_n} P^*_{n-1} \xrightarrow{\wedge^{r_n} f_n{}^*} \bigwedge^{r_n} P_n{}^* = R.$$

Since depth $I(f_n) \geqslant 2$ and the complex above is depth-sensitive, the (dual) sequence

$$0 \to R = \bigwedge^{r_n} P_n \xrightarrow{\wedge^{r_n} f_n} \bigwedge^{r_n} P_{n-1} \to \bigwedge^{r_n+1} P_{n-1} \otimes P_n{}^*$$

is exact.

Our next observation is that the composition

$$P_n \otimes \bigwedge^{t-1} P_{n-1} \to \bigwedge^{t} P_{n-1} \xrightarrow{\wedge^t f_{n-1}} \bigwedge^{t} P_{n-2}$$

is zero for any $t \geqslant 1$, where the map $P_n \otimes \wedge^{t-1} P_{n-1} \to \wedge^t P_{n-1}$ is defined by $a \otimes b \to f_n(a) \wedge b$. Thus the dual sequence

$$\bigwedge^{t} P^*_{n-2} \xrightarrow{\wedge^t f^*_{n-1}} \bigwedge^{t} P^*_{n-1} \to \bigwedge^{t-1} P^*_{n-1} \otimes P_n{}^*$$

also has composition zero.

Letting $t = r_{n-1}$, and making the identifications:

$$\bigwedge^{r_n} P_{n-1} = \bigwedge^{r_{n-1}} P^*_{n-1}; \qquad \bigwedge^{r_n+1} P_{n-1} = \bigwedge^{r_{n-1}-1} P^*_{n-1},$$

we can show that the following diagram is commutative:

$$\begin{tikzcd}
\bigwedge^{r_{n-1}} P^*_{n-2} \arrow[r, "\wedge^{r_{n-1}} f_{n-1}"] & \bigwedge^{r_{n-1}} P^*_{n-1} \arrow[r] \arrow[d, equal] & \bigwedge^{r_{n-1}-1} P^*_{n-1} \otimes P_n{}^* \arrow[d, equal] \\
0 \to R = \bigwedge^{r_n} P_n \arrow[r, "\wedge^{r_n} f_n"] & \bigwedge^{r_n} P_{n-1} \arrow[r] & \bigwedge^{r_n+1} P_{n-1} \otimes P_n{}^*.
\end{tikzcd}$$

Thus there is a unique map $a_{n-1}^*\colon \wedge^{r_{n-1}} P_{n-2}^* \to R$ making the left square commutative.

We now turn to the general step.

Suppose that $k > 1$ and that a_{k+1} and a_k have been defined so that

$$\begin{tikzcd} \bigwedge^{r_k} P_k \arrow[rr, "\wedge^{r_k} f_k"] \arrow[dr, "a_{k+1}^*"'] & & \bigwedge^{r_k} P_{k-1} \\ & R \arrow[ur, "a_k"'] & \end{tikzcd}$$

commutes.

We will construct a map $d\colon \wedge^{r_k} P_{k-1} \to \wedge^{r_k+1} P_{k-1} \otimes P_k{}^*$ such that the diagram

$$0 \to R \xrightarrow{a_k} \bigwedge^{r_k} P_{k-1} \xrightarrow{d} \bigwedge^{r_k+1} P_{k-1} \otimes P_k{}^* \tag{3.3}$$

is exact, and the composition

$$\bigwedge^{r_{k-1}} P_{k-2}^* \xrightarrow{\wedge^{r_{k-1}} f^*} \bigwedge^{r_{k-1}} P_{k-1}^* = \bigwedge^{r_k} P_{k-1} \xrightarrow{d} \bigwedge^{r_k+1} P_{k-1} \otimes P_k{}^* \tag{3.4}$$

is zero. From this it will follow that there is a unique map a making the following diagram commute:

$$\begin{tikzcd} \bigwedge^{r_{k-1}} P_{k-2}^* \arrow[rr, "\wedge^{r_{k-1}} f_{k-1}^*"] \arrow[dr, "a"'] & & \bigwedge^{r_{k-1}} P_{k-1}^* \\ & R \arrow[ur, "a_k"'] & \end{tikzcd}$$

Clearly the map $a_{k-1} = a^*$ satisfies condition (a) of Theorem 3.1.

In other sections we will need to use maps similar to d, so we will construct now a whole family of such maps and establish two of their properties.

If $f: P \to Q$ is a map of finitely generated projective R-modules, f induces a map

$$Q^* \otimes P \to R$$

and, by dualizing, a map

$$R \xrightarrow{\tilde{f}} Q \otimes P^*.$$

Set $r = \operatorname{rank}(f)$, and let

$$d_i^f: \bigwedge^{r+i} Q \to \bigwedge^{r+i+1} Q \otimes P^* \tag{3.5}$$

be the composite map

$$\bigwedge^{r+i} Q = \bigwedge^{r+i} Q \otimes R \xrightarrow{1 \otimes \tilde{f}} \bigwedge^{r+i} Q \otimes Q \otimes P^* \xrightarrow{m \otimes 1} \bigwedge^{r+i+1} Q \otimes P^*, \tag{3.6}$$

where $m: \wedge^{r+i} Q \otimes Q \to \wedge^{r+i+1} Q$ is the usual multiplication in the exterior algebra $\wedge Q$.

We now collect the facts that we need about the maps d_i^f into a lemma. The proof is postponed until the next section.

LEMMA 3.2. *Let R be a commutative ring, and let $f: P \to Q$ be a homomorphism of finitely generated projective R-modules with* $\operatorname{rank}(f) = r$. *Then for each $i \geqslant 0$:*

(a) *The composition*

$$\bigwedge^{i} Q \otimes \bigwedge^{r} P \xrightarrow{m(1 \otimes \Lambda^r f)} \bigwedge^{r+i} Q \xrightarrow{d_i^f} \bigwedge^{r+i+1} Q \otimes P^* \tag{3.7}$$

is zero;

(b) *if $I(f) = R$, then $I(d_i^f) = R$, and the sequence* (3.7) *is exact*; *hence* $\operatorname{Rad}(I(d_i^f)) \supseteq \operatorname{Rad}(I(f))$.

(c) *If Q is an oriented projective R-module of rank $r + r'$, then the following diagram commutes up to sign*:

$$\begin{array}{ccc} \bigwedge^{r+i} Q & \xrightarrow{\quad d_i^f \quad} & \bigwedge^{r+i+1} Q \otimes P^* \\ \| & & \| \\ \bigwedge^{r'-i} Q^* & \xrightarrow{(f \otimes 1)^* m^*} & \bigwedge^{r'-i-1} Q^* \otimes P^*; \end{array} \tag{3.8}$$

(d) *if* $P \to^f Q \to^g L \to 0$ *is exact, then*

$$0 \to \left(\bigwedge^{r'-i} L\right)^* \xrightarrow{(\wedge^{r'-i} g)^*} \bigwedge^{r'-i} Q^* \xrightarrow{(f\otimes 1)^* m^*} \bigwedge^{r'-i-1} Q^* \otimes P^* \qquad (3.9)$$

is exact.

We defer the proof of this lemma to the next section, and complete the proof of Theorem 3.1.

We set $d = d_0^{f_k} : \wedge^{r_k} P_{k-1} \to \wedge^{r_k+1} P_{k-1} \otimes P_k{}^*$.

We must show that this makes the sequence (3.3) exact and the composition (3.4) zero.

First we note that since, by hypothesis, $\wedge^{r_k} f_k = a_k a_{k+1}^*$, we have $I(f_k) = I(a_{k+1})\, I(a_k)$ and so both $I(a_{k+1})$ and $I(a_k)$ contain $I(f_k)$. Since we are assuming that $k > 1$, we know from Theorem 1.2 that depth $I(f_k) \geqslant 2$.

By Lemma 3.2, we know that the top row of the commutative diagram

$$\begin{tikzcd}
\bigwedge^{r_k} P_k \arrow[rr, "\wedge^{r_k} f"] \arrow[dr, "a_{k+1}^*"'] & & \bigwedge^{r_k} P_{k-1} \arrow[r, "d"] & \bigwedge^{r_k+1} P_{k-1} \otimes P_k{}^* \\
& R \arrow[ur, "a_k"'] & &
\end{tikzcd}$$

does have composition zero. Hence, if we denote by J the image of a_{k+1}^*, we have $J\,\mathrm{Im}(da_k) = 0$. But $J = I(a_{k+1})$ and depth $I(a_{k+1}) \geqslant 2$, so J contains nonzero divisors. Since $\mathrm{Im}(da_k)$ is a submodule of the free module $\wedge^{r_k+1} P_{k-1} \otimes P_k{}^*$, the fact that $J\,\mathrm{Im}(da_k) = 0$ implies that $\mathrm{Im}(da_k) = 0$. Hence $da_k = 0$ and the sequence (3.3) is a complex.

To prove that (3.3) is exact, we apply Corollary 1.3, which tells us that it is sufficient to prove exactness after localizing at an arbitrary prime ideal x such that depth $(xR_x) < 2$. Since $k \geqslant 2$, $I(f_k)$ cannot be contained in such a prime. It thus suffices to prove exactness under the hypothesis

$$R = I(f_k) = I(a_k) = I(a_{k+1}).$$

That a_k is a monomorphism is now obvious, and the exactness of (3.3) at the term $\wedge^{r_k} P_{k-1}$ follows from Lemma 3.2(b) with $f = f_k$, $P = P_k$, $Q = P_{k-1}$, $i = 0$.

Finally, to show that the composition of the maps in (3.4) is zero, we note that with $L = \operatorname{Cok} f_k$, we have the commutative diagram:

$$\begin{tikzcd} P_{k-1} \arrow[rr, "f_{k-1}"] \arrow[dr, "g"'] & & P_{k-2} \\ & L \arrow[ur] & \end{tikzcd}$$

Thus the map $\wedge^{r_{k-1}} f_{k-1}^*$ can be factored:

$$\begin{tikzcd} \bigwedge^{r_{k-1}} P_{k-2}^* \arrow[rr, "\wedge^{r_{k-1}} f_{k-1}^*"] \arrow[dr] & & \bigwedge^{r_{k-1}} P_{k-1}^* \\ & \left(\bigwedge^{r_{k-1}} L\right)^* \arrow[ur] & \end{tikzcd}$$

and it therefore suffices to show that the composite

$$\left(\bigwedge^{r_{k-1}} L\right)^* \longrightarrow \bigwedge^{r_{k-1}} P_{k-1}^* = \bigwedge^{r_k} P_{k-1} \xrightarrow{d} \bigwedge^{r_k+1} P_{k-1} \otimes P_k^*$$

is zero. But Lemma 3.2(c) and (d), applied to the case $P = P_k$, $Q = P_{k-1}$, $f = f_k$ and $i = 0$, give us the following diagram, which commutes up to sign, with exact lower row:

$$\begin{tikzcd} & & \bigwedge^{r_k} P_{k-1} \arrow[r, "d"] \arrow[d, equal] & \bigwedge^{r_k+1} P_{k-1} \otimes P_k^* \arrow[d, equal] \\ 0 \arrow[r] & \left(\bigwedge^{r_{k-1}} L\right)^* \arrow[r] & \bigwedge^{r_{k-1}} P_{k-1}^* \arrow[r] & \bigwedge^{r_{k-1}-1} P_{k-1}^* \otimes P_k^*. \end{tikzcd}$$

Hence the composition in (3.10) is zero and this completes the proof of part (a) of Theorem 3.1.

Proof of (b). By part (a) of Theorem 3.1, we have $I(a_{k+1})\, I(a_k) = I(f_k)$. Thus $I(a_k) \supseteq I(f_k))$ for every k, and so it suffices to show that, for $k > 1$, $\operatorname{Rad}(I(a_k)) \subseteq \operatorname{Rad}(I(f_k))$.

Suppose this were not so. Then there is a prime ideal x which is minimal over $I(f_k)$ but which does not contain $I(a_k)$. Localizing at this prime, we may assume that R is local, $I(a_k) = R$, and $I(f_k)$ is primary to the maximal ideal of R.

Since R is local, P_{k-1} is free; let $p_1, \ldots, p_m$ be a basis. Since a_k is a map from R to $\wedge^{r_k} P_{k-1}$, we may write $a_k(1) = \sum a_k^{i_1 \cdots i_r} p_{i_1} \wedge \cdots \wedge p_{i_r}$, where we have set $r = r_k$. Since $I(a_k) = R$ and R is local, one of the elements $a_k^{i_1 \cdots i_r}$ must be a unit. Renumbering the basis elements if necessary, we may assume that $a_k^{1 \cdots r}$ is a unit.

Let P be the summand of P_{k-1} generated by the first r basis elements of P_{k-1}, and let $g\colon P_{k-1} \to P$ be the projection. If we set $f = gf_k$, then clearly $\operatorname{rank}(f) = \operatorname{rank}(f_k) = r$ and $I(f) = I(a_{k+1})\, a_k^{1 \cdots r} = I(a_{k+1})$. Thus the sequence

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \xrightarrow{f_{k+1}} P_k \xrightarrow{f} P$$

is exact by Theorem 1.2.

Now $\operatorname{Cok}(f)$ obviously has finite homological dimension and, since $I(f_k) \neq R$, $hd(\operatorname{Cok} f) = hd(\operatorname{Cok} f_1) - k + 1 < hd(\operatorname{Cok} f_1)$ since $k > 1$. Thus $hd(\operatorname{Cok} f)$ is strictly less than the depth d, of the maximal ideal of R. It is known [A-B 1] that if M is an R-module and $hdM < \infty$, then $hdM = d$ if and only if the maximal ideal of R is an associated ideal of the annihilator of M. Now $I(f) = I(a_{k+1}) = I(f_k)$, so $I(f)$ is primary to the maximal ideal of R. Moreover, $I(f)$ is contained in the annihilator of $\operatorname{Cok}(f)$, since $\operatorname{rank}(f) = \operatorname{rank} P$. Thus, $hd(\operatorname{Cok} f) = d$ which is a contradiction. This proves Theorem 3.1, modulo Lemma 3.2.

We shall prove Lemma 3.2 in the next section. However, before proceeding with this proof, we give an example to show that Theorem 3.1 may fail for resolutions that are not finite.

Let K be any field, let $S = K[X, Y]_{(X,Y)}$ and let $R = S/(Y^2 - X^3)$. We will make the convention that the free R-module R^n comes equipped with a basis, whose elements are $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$, and we shall write $\bar{X}$, $\bar{Y}$ for the cosets of X and Y in R. The R-module $R/(\bar{X}, \bar{Y})$, which is isomorphic to K, has a minimal resolution that begins as follows:

$$\cdots \xrightarrow{f_4} R^2 \xrightarrow{f_3} R^2 \xrightarrow{f_2} R^2 \xrightarrow{f_1} R,$$

where f_1 is the matrix $(\bar{X}, \bar{Y})$ and

$$f_k = \begin{pmatrix} \bar{Y} & \bar{X}^2 \\ -\bar{X} & -\bar{Y} \end{pmatrix}$$

for all $k > 1$. Obviously $\operatorname{rank}(f_k) = 1$ for all $k > 1$.

If there were maps a_k for this resolution making the conclusion of Theorem 3.1 true, then we would have a commutative diagram:

$$\begin{tikzcd} R^2 \arrow[rr, "(\bar{X},\bar{Y})"] \arrow[dr, "a_2^*"'] & & R \\ & R \arrow[ur, "a_1"'] & \end{tikzcd}$$

Thus $a_1 = 1$ and $a_2^* = (\bar{X}, \bar{Y})$. We would also have a commutative diagram:

$$\begin{tikzcd} R^2 \arrow[rr, "f_2"] \arrow[dr, "a_3^*"'] & & R^2 \\ & R \arrow[ur, "a_2=\binom{\bar{X}}{\bar{Y}}"'] & \end{tikzcd}$$

Thus we would have to have:

$$f_2(1, 0) = (\bar{Y}, -\bar{X}) = a_2 a_3^*(1, 0) = (a_3^*(1, 0))(\bar{X}, \bar{Y}).$$

In particular, $\bar{Y} = \bar{X} a_3^*(1, 0)$ and this is impossible in the ring R.

Remark. If the modules P_k of the complex (3.1) were only assumed to be projective of well-defined rank, we could modify Theorem 3.1 and its proof as follows. Let $s_k = \operatorname{rank} P_k$. Since a projective module cannot, in general, be oriented, the proof of Theorem 3.1 must be modified by replacing the identifications

$$\bigwedge^{r_{k+1}} P_k = \bigwedge^{r_k} P_k^*$$

by the natural isomorphisms

$$n\colon \bigwedge^{s_k} P_k^* \otimes \bigwedge^{r_{k+1}} P_k \to \bigwedge^{r_k} P_k^*.$$

Define $R_n = \wedge^{s_n} P_n$ and $R_k = \wedge^{s_k} P_k \otimes R_{k+1}^*$. The modified proof yields the existence of unique maps

$$a_k\colon R_k \to \bigwedge^{r_k} P_{k-1}$$

such that

(i) $a_n = \wedge^{r_n} f_n$

(ii) The following diagrams commute:

$$\begin{tikzcd}
\bigwedge^{r_k} P_k \arrow[rr, "\wedge^{r_k} f_k"] \arrow[d, "n^*"'] & & \bigwedge^{r_k} P_{k-1} \\
\bigwedge^{s_k} P_k \otimes \bigwedge^{r_{k+1}} P_k^* \arrow[dr, "1\otimes a_{k+1}^*"'] & & \\
& R_k \arrow[uur, "a_k"'] &
\end{tikzcd}$$

4. A Lemma from Linear Algebra

Throughout this section, R will be a commutative, but not necessarily noetherian, ring.

Let $f: P \to Q$ be an R-homomorphism. Then f induces algebra maps $\wedge f: \wedge P \to \wedge Q$ and $\wedge f^*: \wedge Q^* \to \wedge P^*$, as well as maps $(\wedge^k Q)^* \otimes \wedge^k P \to R$ for every $k \geqslant 0$. The natural map $\wedge^k Q^* \to (\wedge^k Q)^*$ therefore gives us maps $\wedge^k Q^* \otimes \wedge^k P \to^{f^{(k)}} R$ for every $k \geqslant 0$. Since $\wedge P$ is a graded Hopf algebra, the diagonal map $\Delta: \wedge P \to \wedge P \otimes \wedge P$ maps $\wedge^k P$ into $\sum_s \wedge^s P \otimes \wedge^{k-s} P$ so we have maps $\Delta_{k,s}: \wedge^k P \to \wedge^s P \otimes \wedge^{k-s} P$. Using these maps we may define a $\wedge Q^*$-module structure on $\wedge P$ by setting $\wedge^s Q^* \otimes \wedge^k P \to \wedge^{k-s} P$ to be the composition

$$\bigwedge^{s} Q^* \otimes \bigwedge^{k} P \xrightarrow{1\otimes\Delta_{k,s}} \bigwedge^{s} Q^* \otimes \bigwedge^{s} P \otimes \bigwedge^{k-s} P \xrightarrow{f^{(s)}\otimes 1} R \otimes \bigwedge^{k-s} P = \bigwedge^{k-s} P.$$

This map is zero if $k < s$ and is just the map $f^{(s)}$ when $k = s$.

Using the identity map from P to P, we can in this way define a $\wedge P^*$-module structure on $\wedge P$. The algebra map $\wedge Q^* \to^{\wedge f^*} \wedge P^*$ can then be used to make $\wedge P^*$ a $\wedge Q^*$-module, and it is easy to see that this yields the same $\wedge Q^*$-module structure as that defined in the preceding paragraph.

A similar discussion shows that $\wedge P^*$ is a $\wedge P$-module and $\wedge Q^*$ is a $\wedge P$-module.

If P is an oriented free R-module of rank p, and $e \in \wedge^p P$ is the orientation, then the identification of $\wedge^k P^*$ with $\wedge^{p-k} P$ is given by sending $\beta \in \wedge^k P^*$ to the element $\beta(e) \in \wedge^{p-k} P$, where $\beta(e)$ denotes the operation of β on the element e.

In order to prove Lemma 3.2, we shall make use of two lemmas which relate the various maps and structures defined above. The first lemma is the statement that the Hopf algebra $\wedge Q^*$ measures the algebra $\wedge Q$.

Lemma 4.1. *Writing $m\colon \wedge Q \otimes \wedge Q \to \wedge Q$ for the multiplication in $\wedge Q$, and $n\colon \wedge Q^* \otimes \wedge Q \to \wedge Q$ for the $\wedge Q^*$-module structure of $\wedge Q$, the following diagram is commutative:*

$$\begin{tikzcd}
\wedge Q^* \otimes \wedge Q \otimes \wedge Q \arrow[rr, "1\otimes m"] \arrow[d, "\Delta\otimes 1\otimes 1"] & & \wedge Q^* \otimes \wedge Q \arrow[ddd, "n"] \\
\wedge Q^* \otimes \wedge Q^* \otimes \wedge Q \otimes \wedge Q \arrow[d, equal] & & \\
\wedge Q^* \otimes \wedge Q \otimes \wedge Q^* \otimes \wedge Q \arrow[dr, "n\otimes n"'] & & \\
& \wedge Q \otimes \wedge Q \arrow[r, "m"] & \wedge Q
\end{tikzcd}$$

where the isomorphism on the left is given by the commutativity of the tensor product of graded algebras (there is, of course, a sign involved).

A proof of this lemma may be found in [6].

Lemma 4.2. *With the above notation, the following diagram commutes:*

$$\begin{tikzcd}
\wedge Q^* \otimes \wedge Q \arrow[r, "n"] \arrow[d, "1\otimes\Delta"] & \wedge Q \arrow[dd, "\Delta"] \\
\wedge Q^* \otimes \wedge Q \otimes \wedge Q \arrow[d, equal] & \\
\wedge Q \otimes \wedge Q^* \otimes \wedge Q \arrow[r, "1\otimes n"] & \wedge Q \otimes \wedge Q
\end{tikzcd}$$

where again the isomorphism on the left is given by commutativity of the tensor product.

Proof. In terms of elements, the commutativity of the above diagram may be interpreted as

$$\Delta(t(q)) = \sum_i \pm q_i \otimes t(q_i') \tag{4.1}$$

for $t \in \wedge Q^*$, $q \in \wedge Q$, and $\sum_i q_i \otimes q_i' = \Delta(q)$.

If degree $(q) = 1$, i.e., $q \in \wedge^1 Q = Q$, then $\Delta(q) = (q \otimes 1) + (1 \otimes q)$ and (4.1) is clearly true since, in that case, $t(q) = 0$ unless $t \in \wedge^0 Q^*$ or $t \in \wedge^1 Q^*$. (This is the only case we will use in this paper.)

We now assume that (4.1) is true if $q \in \wedge^s Q$, and prove it for $q \in \wedge^{s+1} Q$. It suffices to do this when $q = q_1 \wedge q_2$ with $q_1 \in Q$ and $q_2 \in \wedge^s Q$. Since $\Delta: \wedge Q \to \wedge Q \otimes \wedge Q$ is an algebra map, we have

$$\Delta(q) = \Delta(q_1)\,\Delta(q_2) = ((q_1 \otimes 1) + (1 \otimes q_1))\,\Delta(q_2) \tag{4.2}$$

We also have, by Lemma 4.1, that

$$t(q) = t(q_1 \wedge q_2) = \pm q_1 \wedge t(q_2) + \sum_j \pm t_j(q_1) \wedge t_j'(q_2)$$

where $\Sigma\, t_j \otimes t_j'$ is the projection of $\Delta(t)$ into $Q^* \otimes \wedge^{d-1} Q^*$, where $d =$ degree (t). Since t_j and q_1 have degree 1, $t_j(q_1)$ is a scalar, and we have

$$\Delta(t(q)) = \pm \Delta(q_1)\,\Delta(t(q_2)) + \sum \pm t_j(q_1)\,\Delta(t_j'(q_2)).$$

Hence, letting $\Delta(q_2) = \sum q_{2i} \otimes q_{2i}'$ and using our induction assumption, we have

$$\Delta(t(q)) = \pm ((q_1 \otimes 1) + (1 \otimes q_1)) \left(\sum q_{2i} \otimes t(q_{2i}')\right)$$
$$+ \sum \pm t_j(q_1)(q_{2i} \otimes t_j'(q_{2i}')).$$

Since $\Sigma \pm t_j(q_1)(q_{2i} \otimes t_j'(q_{2i}')) = \Sigma \pm q_{2i} \otimes (t(q_1 \wedge q_{2i}')) \pm q_1 \wedge t(q_{2i}')$, we have $\Delta(t(q)) = \Sigma \pm q_1 \wedge q_{2i} \otimes t(q_{2i}') + \Sigma \pm q_{2i} \otimes t(q_1 \wedge q_{2i}')$. But, by (4.2) we have $\Delta(q) = \Sigma \pm q_1 \wedge q_{2i} \otimes q_{2i}' + \Sigma \pm q_{2i} \otimes q_1 \wedge q_{2i}'$ so that the above formula for $\Delta(t(q))$ is the one asserted in Lemma 4.2.

With these preliminary lemmas at our disposal, we are ready to prove Lemma 3.2. We shall therefore assume from now on that P and Q are oriented free R-modules and that $f: P \to Q$ is a homomorphism of rank r.

Recall that there were four parts to that lemma. Part (a) was the statement that the composition

$$\overset{i}{\wedge} Q \otimes \overset{r}{\wedge} P \xrightarrow{m(1 \otimes \wedge^r f)} \overset{r+i}{\wedge} Q \xrightarrow{d_i^f} \overset{r+i+1}{\wedge} Q \otimes P^* \tag{4.3}$$

is zero.

Proof of (a). Since P and Q are free, we may dualize, and prove instead that the composition of the dual sequence:

$$\bigwedge^{r+i+1} Q^* \otimes P \xrightarrow{(d_i^f)^*} \bigwedge^{r+i} Q^* \xrightarrow{(1\otimes\wedge^r f^*)m^*} \bigwedge^{i} Q^* \otimes \bigwedge^{r} P^*$$

is zero. This is convenient because $(d_i^f)^*$ is just the map associated to the $\wedge P$-module structure on $\wedge Q^*$ induced by $\Lambda f\colon \wedge P \to \wedge Q$, and $(1 \otimes \wedge^r f^*)\, m^*$ is a component of the map

$$\bigwedge Q^* \xrightarrow{\Delta} \bigwedge Q^* \otimes \bigwedge Q^* \xrightarrow{1\otimes\wedge f^*} \bigwedge Q^* \otimes \bigwedge P^*.$$

Let $p \in P$, $q \in \wedge^{r+i+1} Q^*$. We want to show that

$$\left(1 \otimes \bigwedge^{r} f^*\right)(\Delta(f(p)(q))) = 0,$$

where we write $f(p)(q)$ for the result of acting on q with $(f(p)$. By Lemma 4.2, we have, setting $\Delta(q) = \sum q_i \otimes q_i'$,

$$\Delta(f(p)(q)) = \sum q_i \otimes f(p)(q_i')$$

and

$$\begin{aligned}\left(1 \otimes \bigwedge^{r} f^*\right)(\Delta(f(p)(q))) &= \sum q_i \otimes \bigwedge^{r} f^*(f(p)(q_i')) \\ &= \sum q_i \otimes f(p)\left(\bigwedge^{r+1} f^*(q_i')\right) \qquad (4.4)\end{aligned}$$

since $\wedge f^*$ is a map of $\wedge P$-modules. However, the last expression of (4.4) is zero since $\wedge^{r+1} f^* = 0$ by hypothesis.

Proof of (b). Part (b) of Lemma 3.2 asserts that if $I(f) = R$, then $I(d_i^f) = R$ and the sequence (4.3) is exact.

By Lemma 1.1, $I(f) = R$ implies that $\operatorname{Cok} f$ is projective. Setting $Q'' = \operatorname{Cok} f$, $Q' = \operatorname{Im} f$, and $P' = \ker f$, we have $Q = Q' \oplus Q''$, $P = P' \oplus Q'$, and the map f is the composite

$$P = P' \oplus Q' \xrightarrow{\text{projection}} Q' \xrightarrow{\text{inclusion}} Q' \oplus Q'' = Q.$$

Moreover, we have $\operatorname{rank} Q' = r$.

Using the fact that $\wedge(A \oplus B) = \wedge A \otimes \wedge B$, we see that the sequence (4.3) may be analyzed as follows:

$$\Big(\sum_{i=k+l} \overset{k}{\wedge} Q'' \otimes \overset{l}{\wedge} Q'\Big) \otimes \Big(\sum_{r=k+l} \overset{k}{\wedge} P' \otimes \overset{l}{\wedge} Q'\Big)$$

$$\xrightarrow{g} \sum_{k+l=r+i} \overset{k}{\wedge} Q'' \otimes \overset{l}{\wedge} Q'$$

$$\xrightarrow{h} \Big(\sum_{k+l=r+i+1} \overset{k}{\wedge} Q'' \otimes \overset{l}{\wedge} Q'\Big) \otimes (Q'^* \otimes P'^*),$$

where $g = m(1 \otimes \wedge^r f)$ and $h = d_i^f$.

We first examine the image of $m(1 \otimes \wedge^r f)$. Clearly this map is the direct sum of the maps

$$\overset{k}{\wedge} Q'' \otimes \overset{l}{\wedge} Q' \otimes \overset{s}{\wedge} P' \otimes \overset{t}{\wedge} Q' \to \overset{u}{\wedge} Q'' \otimes \overset{v}{\wedge} Q',$$

where $k + l = i$, $s + t = r$, and $u + v = r + i$. But these maps are zero for $s > 0$ because $f: P' \to Q$ is zero. Thus $m(1 \otimes \wedge^r f)$ has the same image as the sum of the maps

$$\overset{k}{\wedge} Q'' \otimes \overset{l}{\wedge} Q' \otimes \overset{r}{\wedge} Q' \to \overset{k}{\wedge} Q'' \otimes \overset{l+r}{\wedge} Q',$$

where $k + l = i$. However, since rank $Q' = r$, $\wedge^{l+r} Q' = 0$ for $l > 0$, so we need only look at the term

$$\overset{i}{\wedge} Q'' \otimes \overset{r}{\wedge} Q' \to \overset{i}{\wedge} Q'' \otimes \overset{r}{\wedge} Q'.$$

Thus the image of $m(1 \otimes \wedge^r f)$ is

$$\overset{i}{\wedge} Q'' \otimes \overset{r}{\wedge} Q' \subset \sum_{k+l=r+i} \overset{k}{\wedge} Q'' \otimes \overset{l}{\wedge} Q'.$$

In order to analyze ker d_i^f, we will dualize and examine the image of

$$(d_i^f)^*: \sum_{k+l=r+i+1} \overset{k}{\wedge} Q''^* \otimes \overset{l}{\wedge} Q'^* \otimes (Q' \otimes P') \to \sum_{k+l=r+i} \overset{k}{\wedge} Q''^* \otimes \overset{l}{\wedge} Q'^*. \tag{4.5}$$

As we have seen, $(d_i^f)^*$ is induced by the action of $P = Q' \oplus P'$ on $\wedge Q^*$. Thus $(d_i^f)^*$ is the direct sum of the maps

$$\bigwedge^k Q''^* \otimes \bigwedge^l Q'^* \otimes (Q' \oplus P') \to \bigwedge^k Q''^* \otimes \bigwedge^l Q'^*$$

for $k + l = r + i$, induced by action of Q' on $\wedge Q'^*$. These maps are clearly onto, because the maps

$$\bigwedge^l Q'^* \otimes Q' \to \bigwedge^{l-1} Q'^*$$

are onto for all $l \leqslant r = \operatorname{rank} Q'$. Thus the image of $(d_i^f)^*$ is

$$\sum_{k,l} \bigwedge^k Q''^* \otimes \bigwedge^{l-1} Q',$$

where $k + l = r + i + 1$, and $\operatorname{Cok}(d_i^f)^* = \wedge^i Q''^* \otimes \wedge^r Q'^*$. Thus $\ker d_i^f = \wedge^i Q'' \otimes \wedge^r Q'$ as required for the exactness of (4.3). Moreover, $\operatorname{Cok}(d_i^f)^*$ is projective, of constant rank, so by Lemma 1.1 $I((d_i^f)^*) = R$. This completes the proof of (b).

Proof of (c). To prove part (c), we must prove the commutativity (up to sign) of the diagram:

$$\begin{array}{ccc}
\bigwedge^{r+i} Q & \xrightarrow{\;d_i^f\;} & \bigwedge^{r+i+1} Q \otimes P^* \\
\| & & \| \\
\bigwedge^{r'-i} Q^* & \xrightarrow{(f^*\otimes 1)m^*} & \bigwedge^{r'-i-1} Q^* \otimes P^*,
\end{array}$$

where $r + r' = \operatorname{rank} Q$.

Here we use the standard identification

$$\bigwedge^k Q^* \to \bigwedge^{r+r'-k} Q,$$

which is as follows: Recall that if we let $e \in \wedge^{r+r'} Q$ be the generator of $\wedge^{r+r'} Q$, which determines the orientation of Q, then the isomorphism $\wedge^k Q^* \to \wedge^{r+r'-k} Q$ is given by $q \to q(e)$ for $q \in \wedge^k Q^*$.

It is more convenient to prove that the dual of (4.5) commutes up to sign. Thus we must show that for $q \in \wedge^{r+i+1} Q^*$, $p \in P$, we have

$$(p(q))(e) = f(p) \wedge q(e).$$

Of course, by $p(q)$ we mean $f(p)(q)$ since this is the action of P on $\wedge Q^*$. Letting $s = f(p)$ and $\Delta(q) = \sum_i q_i \otimes q_i'$, we have by Lemma 4.1,

$$q(s \wedge e) = \sum_i q_i(s) \wedge q_i'(e).$$

Since $s \wedge e \in \wedge^{r+r'+1} Q = 0$, and since $q_i(s) = 0$ if $\mathrm{degree}(q_i) > 1$, we have

$$0 = s \wedge q(e) + \sum_{\deg(q_i)=1} q_i(s) \wedge q_i'(e).$$

Since $q_i(s) = s(q_i)$ when $\deg(q_i) = 1$, we get

$$s \wedge q(e) = -\sum_i s(q_i)\, q_i'(e) = \left(-\sum s(q_i)\, q_i'\right)(e) = (-s(q))(e)$$

as required.

Proof of (d). Finally, we must prove that if $P \to^f Q \to^g L \to 0$ is exact, then the sequence

$$0 \to \left(\bigwedge^{r'-i} L\right)^* \xrightarrow{(\wedge^{r'-i} g)^*} \bigwedge^{r'-i} Q^* \xrightarrow{(f\otimes 1)^* m^*} \bigwedge^{r'-i-1} Q^* \otimes P^* \qquad (4.6)$$

is exact.

This follows by dualization from the exactness of the sequence

$$\bigwedge^{r'-i-1} Q \otimes P \to \bigwedge^{r'-i} Q \to \bigwedge^{r'-i} L \to 0 \qquad (4.7)$$

whose exactness is standard (see [4, p. 78, Prop. 3]). This concludes the proof of Lemma 3.2.

5. Some Applications of the Structure Theorem

In this section we discuss some applications of Theorem 3.1. Part of this material appeared in [7].

We will fix our attention on the following situation. Let R be a noetherian ring, and let

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0$$

be an exact sequence of finitely generated free R-modules such that $\mathrm{Cok}\, f_1$ is torsion (that is, annihilated by some element, and therefore [2] by some nonzero divisor, of R). In this case, if we choose orientations for the P_i so that we can apply Theorem 3.1, and use the fact that rank $f_1 = r_1 =$ rank P_0, we see that $\wedge^{r_1} P_0 = R$, and

$$a_1 \colon R \to \bigwedge^{r_1} P_0 = R$$

is given by multiplication by some element of R, which we will denote by a. Since $I(f_1) \subseteq I(a_1) = (a)$, and since by Theorem 1.2, $I(f_1)$ contains a nonzero divisor, it follows that a is a nonzero divisor.

COROLLARY 5.1. *With the above notation, the ideals*

$$I(f_1) \cdot I(f_3) \cdot I(f_5) \ldots$$

and

$$I(f_2) \cdot I(f_4) \cdot I(f_6) \ldots$$

are isomorphic.

Proof. The first product may, by virtue of Theorem 3.1, be written

$$(I(a_1)\, I(a_2))(I(a_3)\, I(a_4)) \ldots,$$

while the second product is

$$(I(a_2)\, I(a_3))(I(a_4)\, I(a_5)) \ldots .$$

Clearly, the two products differ by multiplication by a, which is an isomorphism because a is a nonzero divisor.

The next corollary is the main result of [21], where it was deduced in quite a different way.

COROLLARY 5.2. *Let R be a noetherian ring, and let I be an ideal of R. If I admits a finite free resolution, then there exists an ideal I' and a nonzero divisor a of R such that*

$$I = aI'$$

and depth $I' \geqslant 2$.

Proof. Let $0 \to P_n \to^{f_n} P_{n-1} \to \cdots \to P_1 \to^{f_1} P_0 \to R/I \to 0$ be the free resolution of R/I. Let

$$I' = I(a_2)$$

and let a be the generator of $I(a_1)$, where a_2 and a_1 are the maps defined by Theorem 3.1. We have already remarked that a must be a nonzero divisor. The inequality depth $I' \geqslant 2$ follows from Theorem 1.2(b) and the fact that

$$I(a_2) \supseteq I(f_2).$$

Remark. Let I be an ideal of R of finite projective dimension, and let

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to R \to R/I \to 0$$

be a projective resolution of R/I with each P_k free for $k < n$. The remark at the end of Section 3 can be used to show that, if rank $P_n = s$, then

$$\bigwedge^s P_n \cong R$$

if and only if I is isomorphic to an ideal of depth $\geqslant 2$.

We note that Corollary 5.2 has the following consequence.

COROLLARY 5.3 [3]. *Let R be a noetherian ring. Then R is factorial if and only if every ideal of R which can be generated by two elements has a finite free resolution. In particular, every regular local ring is factorial.*

Proof. If R is factorial, then any two elements $s, t \in R$ have a greatest common divisor d, and it is easy to see that $(s/d, t/d)$ form an R sequence. Thus $(s/d, t/d)$ has a finite free resolution, and

$$(s, t) \cong (s/d, t/d)$$

via multiplication by d.

Conversely, if the ideal (s, t) admits a free resolution, then by Corollary 5.2, $(s, t) = aI'$, where $I' = (a^{-1}s, a^{-1}t)$ is of depth 2. Now suppose that s is an irreducible element of R; we wish to show that (s) is a prime ideal. For this it is enough to show that if $t \notin (s)$, then (s, t) has depth 2. We have already seen that $(a^{-1}s, a^{-1}t)$ has depth 2 for some $a \in R$. Since s is irreducible, $a^{-1}s \in R$ implies that a is a unit of R. Thus (s, t) has depth 2 as required.

Theorem 3.1 also gives a necessary condition for the cokernel of a map of free modules to have finite homological dimension. We give an illustration:

Let k be a field, and let $0 < m \leqslant n$ be integers. Set $R = k[X_{ij}]$, the ring of polynomials in mn variables, and let g be the $m \times n$ matrix over R

whose (i,j)-th entry is X_{ij}. We call g the generic $m \times n$ matrix. By remark 3 after Theorem 1.2, depth $I(g) = n - m + 1 = hd(\operatorname{Cok} g)$.

Now fix an integer $p < m$, and let $S = R/I_{p+1}(g)$. Let f be the reduction, modulo $I_{p+1}(g)$, of g; we call f the generic $m \times n$ matrix of rank p.

COROLLARY 5.4. *With the above notation,*

$$hd_S(\operatorname{Cok}(f)) = \infty$$

Proof. Since $I_{p+1}(g)$ is a graded ideal and g is a homogeneous map, S inherits from R the structure of a graded ring, and $\operatorname{Cok}(f)$ is a graded S-module. It follows that if $hd_S(\operatorname{Cok}(f)) < \infty$, then $\operatorname{Cok}(f)$ has a finite free resolution, to which we may apply Theorem 3.1. In this way we obtain a factorization

$$\begin{tikzcd} \overset{p}{\bigwedge} S^n \arrow[rr, "\wedge^p f"] \arrow[dr, "a_2^*"'] & & \overset{p}{\bigwedge} S^m \\ & S \arrow[ur, "a_1"'] & \end{tikzcd}$$

In particular, if $e_1, \ldots, e_n$ is the canonical basis of S^n, then all the $p \times p$ minors which occur in the first p columns of the matrix of f are contained in the principal ideal $(a_2^*(e_1 \wedge \cdots \wedge e_p))$. By the last statement of Theorem 3.1, this ideal is proper.

On the other hand, the $p \times p$ minors of g all have degree p in the graded ring R, and the ideal $I_{p+1}(g)$ is generated by elements of degree $p + 1$. Thus since the $p \times p$ minors in the first p columns of f are contained in a proper principal ideal of S, the $p \times p$ minors of the first p columns of g must be contained in a proper principal ideal of R. However, these are the minors of a generic $p \times m$ matrix, and thus generate an ideal of depth $m - p + 1 > 1$ in R. This contradiction gives the desired result.

6. THE SECOND STRUCTURE THEOREM—LOWER ORDER MINORS

Theorem 0 may be regarded as giving a complete structure theorem for ideals of homological dimension 1, or, more conveniently, for cyclic torsion modules of homological dimension 2. The reason that Theorem

3.1 fails to give a satisfactory structure theorem for ideals of higher dimension, or for noncyclic torsion modules even in homological dimension 2, is that it does not deal adequately with the lower order minors of the maps in a resolution, that is, with the maps $\wedge^p f_k$, for $p < r_k$, in the notation of Theorem 3.1. For example, the simplest case of a module of homological dimension 2 over a local ring R, which is not analyzable in terms of Theorem 3.1 would be a torsion module M with $hdM = 2$, such that M has 2 generators and 4 relations. The free resolution of M then has the form

$$0 \longrightarrow R^2 \xrightarrow{f_2} R^4 \xrightarrow{f_1} R^2.$$

Theorem 3.1 tells us that there exists a nonzero divisor $a = a_1$ such that the diagram:

$$\begin{tikzcd} \bigwedge^2 R^4 \arrow[r, "\wedge^2 f_1"] \arrow[d, equal] & \bigwedge^2 R^2 = R \\ \bigwedge^2 R^{4*} \arrow[r, "\wedge^2 f_2^*"] & \bigwedge^2 R^{2*} = R \arrow[u, "a"'] \end{tikzcd}$$

commutes. But to have, in this case, a result as nice as Theorem 0, it would be necessary to have some expression for the relations between the elements of the matrices of f_2 and f_1 themselves.

The most obvious result analogous to Theorem 0 would be the existence of some expression for f_1 in terms of f_2. Such a result would imply, in particular, that $I_1(f_2) \supseteq I_1(f_1)$. That this relation is in general false may be seen from the following example:

Let $R = K[X, Y]_{(X,Y)}$, where K is any field. The sequence

$$0 \to R^2 \xrightarrow[\begin{pmatrix} xy^2 & 0 \\ y^3 & x^2 \\ x^2 & x^2 \\ 0 & y \end{pmatrix}]{f_2} R^4 \xrightarrow[\begin{pmatrix} y^2 & -xy & 0 & x^3 \\ x & 0 & -y^2 & x^2y \end{pmatrix}]{f_1} R^2$$

may be easily seen to be exact (use Theorem 1.2 and the fact that $\operatorname{Rad} I(f_1) = \operatorname{Rad} I(f_2) = (x, y)$), but clearly neither of $I_1(f_1)$ and $I_1(f_2)$ contains the other.

Surprisingly enough, some of the lower order minors of maps in a resolution do turn out to be accessible, if the resolution has length at least 3. The result is given by Theorem 6.1. Further theorems on the lower order minors of maps in a resolution will be discussed in Section 7.

THEOREM 6.1. *Let R be a noetherian ring, and let*

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0$$

be an exact sequence of oriented free modules. Let $r_k = \operatorname{rank} f_k$, *and let*

$$a_k\colon R \to \bigwedge^{r_k} P_{k-1}$$

be the maps defined by Theorem 3.1. *Let*

$$a_k'\colon \bigwedge^{r_{k-1}-1} P_{k-1} \to P_{k-1}^*$$

be defined as the composite

$$\begin{tikzcd}
\bigwedge^{r_{k-1}-1} P_{k-1} = R \otimes \bigwedge^{r_k+1} P_{k-1}^* \arrow[r, "a_k \otimes 1"] \arrow[dr, "a_k'"'] & \bigwedge^{r_k} P_{k-1} \otimes \bigwedge^{r_k+1} P_{k-1}^* \arrow[d, "n"] \\
 & P_{k-1}^*
\end{tikzcd}$$

Then for $k \geqslant 2$, there exist maps $b_k\colon P_k{}^* \to \wedge^{r_k-1} P_{k-1}$ making the diagram

$$\begin{tikzcd}
\bigwedge^{r_k-1} P_k \arrow[rr, "\wedge^{r_k-1} f_k"] \arrow[dr, "a_{k+1}'"'] & & \bigwedge^{r_k-1} P_{k-1} \\
 & P_k{}^* \arrow[ur, "b_k"'] &
\end{tikzcd} \tag{6.1}$$

commute.

This immediately gives us the following.

COROLLARY 6.2. *With hypothesis and notation as in* 6.1, *we have*

$$I_{r_k-1}(f_k) \subseteq I(a_{k+1})$$

for all $k \geqslant 2$.

Proof of Theorem 6.1. The proof of this theorem follows the pattern established in the proof of Theorem 3.1. We will show that the sequence

$$P_k \xrightarrow{a'^*_{k+1}} \bigwedge^{r_{k+1}+1} P_k \xrightarrow{d} \bigwedge^{r_{k+1}+2} P_k \otimes P^*_{k+1} \tag{6.2}$$

is exact, where we regard the codomain of a'^*_{k+1} as $\bigwedge^{r_{k+1}+1} P_k$ by means of the identification

$$\bigwedge^{r_k-1} P_k{}^* = \bigwedge^{r_{k+1}+1} P_k\,,$$

and where d is the map $d_1^{f_{k+1}}$ of Lemma 3.2.

We will also show that the composition

$$\bigwedge^{r_k-1} P^*_{k-1} \xrightarrow{\bigwedge^{r_k-1} f_k{}^*} \bigwedge^{r_k-1} P_k{}^* = \bigwedge^{r_{k+1}+1} P_k \xrightarrow{d} \bigwedge^{r_{k+1}+2} P_k \otimes P^*_{k+1} \tag{6.3}$$

is zero. Together, these facts imply that there exists a map b making the diagram

$$\begin{tikzcd} \bigwedge^{r_k-1} P^*_{k-1} \arrow[rr, "\bigwedge^{r_k-1} f_k{}^*"] \arrow[dr, "b"'] & & \bigwedge^{r_k-1} P_k{}^* \\ & P_k \arrow[ur, "a'^*_{k+1}"'] & \end{tikzcd}$$

commute. It is evident that the choice $b_k = b^*$ will make the diagram (6.1) commute.

The idea behind this proof, like the idea in the proof of Theorem 3.1, comes from consideration of a generalized Koszul complex (see [9]). For we know that if $n \geqslant 3$, then the initial segment of the dual of this Koszul complex is exact and looks like

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{g} \bigwedge^{r_n+1} P_{n-1} \xrightarrow{d} \bigwedge^{r_n+2} P_{n-1} \otimes P_n{}^*.$$

The map g turns out to be the map a'^*_n defined above. Thus, we know that (6.2) was exact for $k = n - 1$; the problem is how to push this kind of information further down along the given resolution. The following is a uniform treatment for all $k \geqslant 2$.

We begin by showing that the composition is (6.2) is zero. First we observe that the following diagram commutes:

$$\begin{tikzcd}
P_k \otimes \bigwedge^{r_{k+1}} P_{k+1} \arrow[rr, "1\wedge\Lambda^{r_{k+1}} f_{k+1}"] \arrow[dr, "1\otimes a^*_{k+2}"'] & & \bigwedge^{r_{k+1}+1} P_k \\
& P_k \otimes R = P_k \arrow[ur, "a'^*_{k+1}"'] &
\end{tikzcd}$$

where $1 \wedge \Lambda^{r_{k+1}} f_{k+1}$ denotes the composition: $m(1 \otimes \Lambda^{r_{k+1}} f_{k+1})$.

By Lemma 3.2, the composite

$$P_k \otimes \bigwedge^{r_{k+1}} P_{k+1} \xrightarrow{1\wedge\Lambda^{r_{k+1}+1} f_{k+1}} \bigwedge^{r_{k+1}+1} P_k \xrightarrow{d} \bigwedge^{r_{k+1}+2} P_k \otimes P^*_{k+1} \tag{6.4}$$

is zero. Thus if $M \subseteq P_k$ is the image of $1 \otimes a^*_{k+2}$, we have $da'^*_{k+1}(M) = 0$. Clearly we have

$$M = \operatorname{Im}(a^*_{k+2})\, P_k = I(a_{k+2})\, P_k\,.$$

Thus $I(a_{k+2}) \operatorname{Im}(da'^*_{k+1}) = 0$. Since $I(a_{k+2}) \supseteq I(f_{k+2})$, we have depth $I(a_{k+2}) > 3$. In particular, $I(a_{k+2})$ contains a nonzero divisor. Since $\operatorname{Im}(da'^*_{k+1})$ is a submodule of a projective R-module, it follows that $\operatorname{Im}(da'^*_{k+1}) = 0$, so $da'^*_{k+1} = 0$.

To prove that (6.2) is exact, we will embed it in a complex to which we can apply Theorem 1.2. In fact we will show that the sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots P_{k+1} \xrightarrow{f_{k+1}} P_k \xrightarrow{a'^*_{k+1}} \bigwedge^{r_{k+1}+1} P_k \xrightarrow{d} \bigwedge^{r_{k+1}+2} P_k \otimes P^*_{k+1} \tag{6.5}$$

is exact. First we must show that the composition

$$P_{k+1} \xrightarrow{f_{k+1}} P_k \xrightarrow{a'^*_{k+1}} \bigwedge^{r_{k+1}+1} P_k \tag{6.6}$$

is zero. From the equation

$$a^*_{k+2} a_{k+1} = \bigwedge^{r_{k+1}} f^*_{k+1}$$

it easily follows that

$$\begin{tikzcd}
\bigwedge^{r_{k+1}+r_k-1} P_k^* = P_k \arrow[r, "a'^*_{k+1}"] \arrow[dr, "(\bigwedge^{r_{k+1}} f_{k+1} \wedge 1)^*"'] & \bigwedge^{r_{k+1}+1} P_k = P \otimes \bigwedge^{r_k-1} P_k^* \\
 & \bigwedge^{r_{k+1}} P_{k+1}^* \otimes \bigwedge^{r_k-1} P_k^* \arrow[u, "a^*_{k+1} \otimes 1"']
\end{tikzcd}$$

commutes. However, we have a commutative diagram:

$$\begin{tikzcd}
P_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, equal] & P_k \arrow[d, equal] \\
\bigwedge^{r_{k+1}+r_k} P_k^* \otimes P_{k+1} \arrow[r, "d^{f_{k+1}}_{r_k-1}"] & \bigwedge^{r_{k+1}+r_k-1} P_k^*
\end{tikzcd}$$

and thus

$$P_{k+1} \xrightarrow{f_{k+1}} P_k \xrightarrow{(\bigwedge^{r_{k+1}} f_{k+1} \wedge 1)^*} \bigwedge^{r_{k+1}} P_{k+1}^* \otimes \bigwedge^{r_k-1} P_k^* \tag{6.7}$$

has composition zero by Lemma 3.2(a) (since (6.7) is, after appropriate identification, just the dual of the sequence (3.7) with $f = f_{k+1}$, $r = r_{k+1}$, $i = r_k - 1$). Since, as we have already noted, $I(a_{k+2}^*)$ contains a nonzero divisor, $a_{k+2}^* \otimes 1$ is a monomorphism, and the sequence (6.6) has composition zero too.

Now that we know that (6.5) is a complex, we can apply the exactness criterion of Theorem 1.2. Since by hypothesis $k \geqslant 2$, we have depth $I(f_{k+1}) \geqslant 3$, depth $I(f_{k+2}) \geqslant 4$..., and

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_{k+1} \xrightarrow{f_{k+1}} P_k$$

is exact by hypothesis. Thus it suffices to check that

$$\text{depth}\, I(a'^*_{k+1}) \geqslant 2,$$

$$\text{depth}\, I(d) \geqslant 1,$$

and that condition 1 of Theorem 1.2 is satisfied by the ranks of f_{k+1}, a'^*_{k+1}, and d.

We have already seen that $\text{Rad}(I(a'^*_{k+1})) \supseteq \text{Rad}(I(f_{k+1}))$, and by Lemma 3.2(b), $\text{Rad}(I(d)) \supseteq \text{Rad}(I(f_{k+1}))$. Since depth $I(f_{k+1}) \geqslant k + 1 \geqslant 3$ by Theorem 1.1, the depth conditions are amply satisfied.

As for the rank conditions, it suffices to check these after localizing at the multiplicatively closed set generated by a nonzero divisor contained in $I(f_{k+1})$. Thus we may assume that $I(f_{k+1}) = R$, and in this case Lemma 3.2(b) assures us of the exactness of the duals of (6.6) and of (6.4). Thus the rank conditions are satisfied, so (6.5) is exact.

Finally, we must show that the composition in (6.3) is zero. Once again we follow the pattern of Theorem 3.1. Setting $L = \mathrm{Cok}(f_{k+1})$, we get an exact sequence

$$P_{k+1} \to P_k \to L \to 0.$$

Thus f_k factors through L, and $\Lambda^{r_k-1} f_k{}^*$ factors through $(\Lambda^{r_k-1} L)^*$. By Lemma 3.2(d),

$$0 \to \left(\bigwedge^{r_k-1} L\right)^* \to \bigwedge^{r_k-1} P_k{}^* \to \bigwedge^{r_k-2} P_k{}^* \otimes P_{k+1}^*$$

is exact, and in particular has zero composition. Thus

$$\bigwedge^{r_k-1} P_{k-1}^* \xrightarrow{\Lambda^{r_k-1} f_k{}^*} \bigwedge^{r_k-1} P_k{}^* \xrightarrow{m^*} \bigwedge^{r_k-2} P_k{}^* \otimes P_{k+1}^*$$

has zero composition. Since m^* is, after appropriate identifications, the same as d, the composition in (6.3) is zero too. This concludes the proof of Theorem 6.1.

Remark. It is often difficult to apply Theorem 6.1 to compute the maps b_k for particular, explicitly given resolutions. The next proposition sometimes gives a more manageable definition. Since we will have no occasion to use it in this paper, we will not prove it here.

PROPOSITION 6.3. *With the notation and hypothesis of Theorem* 6.1, *a map* b_k: $P_k{}^* \to \Lambda^{r_k-1} P_{k-1}$ *makes diagram* (6.1) *commutative if and only if it make the following diagram commutative*:

$$\begin{array}{ccc}
\bigwedge^{r_k-1} P_{k-1}^* & \xrightarrow{n^*} & \bigwedge^{r_k} P_{k-1}^* \otimes P_{k-1} \\
\Big\downarrow {\scriptstyle b_k{}^*} & & \Big\downarrow {\scriptstyle a_k{}^* \otimes 1} \\
P_k & \xrightarrow[f_k]{} & P_{k-1}
\end{array} .$$

It is easy to see that a map b_k, making (6.8) commute, exists, even independently of Theorem 6.1, from the fact that

$$f_{k-1}[(a_k{}^* \otimes 1)\, n^*] = 0;$$

the difficult thing is to show that such a map does make diagram (6.1) commute.

Remark. As with Theorem 3.1, a minor modification of Theorem 6.1 is valid for arbitrary finite projective resolutions. See the remark at the end of Section 3 for more details.

7. An Application to 3-Generator Ideals

We will restrict our attention in this section to ideals which can be resolved by free modules. This is automatically the case, for example, for ideals in a local ring, or a polynomial ring.

Just as Theorem 1.2 and Theorem 0 give the structure of ideals of homological dimension 1, Theorems 1.2 and 6.1 allow us to give a structure theorem for 3-generator ideals of finite homological dimension. In the next section, we get a more complete result for 3-generator ideals of homological dimension 2. For convenience, we work with the cyclic module R/I rather than with the ideal I.

Suppose that

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 = R \qquad (7.1)$$

is an exact sequence of oriented stably free modules with $P_0 = R$, $P_1 = R^3$. By Theorem 1.2 we have: (a) depth $I(f_k) \geqslant k$ for $k = 1,\ldots, n$; (b) $r_1 = \operatorname{rank}(f_1) = 1$; (c) $r_2 = \operatorname{rank}(f_2) = 2$; (d) $r_3 = \operatorname{rank}(f_3) = \operatorname{rank} P_2 - 2$.

By Theorems 3.1 and 6.1 there exist maps $a_3'\colon P_2 \to P_2^*$, $b_2\colon P_2^* \to P_1$, $a_2\colon R \to \wedge^2 P_1 = P_1^*$, and $a_1\colon R \to P_0$ such that the following diagrams commute:

$$\begin{tikzcd} P_2 \arrow[rr, "f_2"] \arrow[dr, "a_3'"'] & & P_1 \\ & P_2^* \arrow[ur, "b_2"'] & \end{tikzcd} \qquad \begin{tikzcd} P_1 \arrow[rr, "f_1"] \arrow[dr, "a_2^*"'] & & R = P_0 \\ & R \arrow[ur, "a_1"'] & \end{tikzcd} .$$

With the exact sequence (7.1) we will associate the truncated sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_3 \xrightarrow{f_3} P \qquad (7.3)$$

together with the triple $\{P_1, b_2, a_1\}$.

Conversely, suppose that we are given an exact sequence of oriented free modules such as (7.3) satisfying the conditions

$$\operatorname{depth} I(f_k) \geqslant k \qquad \text{for} \quad k = 3, \ldots, n \tag{7.3a}$$

and

$$r_3 = \operatorname{rank}(f_3) = \operatorname{rank} P_2 - 2. \tag{7.3b}$$

Suppose we are also given an oriented free module P_1 of rank 3, and maps:

$$b\colon P_2{}^* \to P_1\,,$$
$$a\colon R \to R.$$

We will regard the map a as an element of R.

Let a_3, $a_3{}'$ be the maps constructed in Theorem 6.1 for the exact sequence (7.3). Using these maps, we can define maps:

$$f_2\colon P_2 \to P_1\,,$$
$$f_1\colon P_1 \to P_0 = R,$$

and

$$f_1{}'\colon P_1 \to R$$

by the following diagrams:

$$\begin{tikzcd}
P_2 \arrow[rr, "f_2"] \arrow[dr, "a_3{}'"'] & & P_1 \\
& P_2{}^* \arrow[ur, "b"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
P_1 \arrow[rr, "f_1"] \arrow[d, equal] \arrow[drr, "f_1{}'"] & & R \\
\overset{2}{\bigwedge} P_1{}^* \arrow[dr, "\Lambda^2 b^*"'] & & R \arrow[u, "a"'] \\
& \overset{2}{\bigwedge} P_2 = \overset{r_3}{\bigwedge} P_2{}^* \arrow[ur, "a_3{}^*"'] &
\end{tikzcd} \tag{7.5}$$

To the sequence (7.3) satisfying conditions (7.3a), (7.3b), and the triple of data: $\{P_1, b, a\}$, we can associate the sequence

$$P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} R. \tag{7.6}$$

THEOREM 7.1. *The sequence* (7.6) *is exact if and only if the element a is not a zero divisor and the image of the map f_1', defined by* (7.5), *has depth* $\geqslant 2$.

From Theorem 7.1, we see that if we start with an exact sequence (7.1), the truncated sequence (7.3) satisfies conditions (7.3a) and (7.3b) and the triple $\{P_1, b_2, a_1\}$ satisfies the conditions of Theorem 7.1. Conversely, if we start with a sequence (7.3) satisfying (7.3a) and (7.3b), and if we are given a triple $\{P_1, b, a\}$ satisfying the conditions of Theorem 7.1, then the sequence (7.6) spliced with (7.3) gives an exact sequence (7.1). We have

THEOREM 7.2. *The correspondences described above between exact sequences* (7.1) *and pairs consisting of an exact sequence* (7.3) *satisfying* (7.3a) *and* (7.3b) *and a triple $\{P_1, b, a\}$ satisfying the conditions of Theorem* 7.1, *are inverses of each other.*

Proof of Theorem 7.1. Assume first that the sequence (7.6) is exact. Splicing it with the exact sequence (7.3), we obtain an exact sequence (7.1). We have already seen that to the exact sequence (7.1) there corresponds the triple $\{P_1, b_2, a_1\}$ and, from the commutativity of (7.2) and (7.4) we may assume that our given triple $\{P_1, b, a\}$ is the triple $\{P_1, b_2, a_1\}$ associated to the exact sequence (7.1). Hence we know that $a = a_1$ is a nonzero divisor. What remains to be shown is that the map f_1' (defined in (7.5)) has as its image an ideal of depth $\geqslant 2$.

We will show that f_1' is the map $a_2{}^*$ defined for the sequence (7.1) by Theorem 3.1. Since we know that depth $I(a_2{}^*) \geqslant 2$, we obtain our result. To see that $f_1' = a_2{}^*$, recall first that the map a_2 is by definition, the unique map making the diagram

$$\begin{tikzcd} \bigwedge^2 P_2 \arrow[rr, "\wedge^2 f_2"] \arrow[dr, "a_3{}^*"'] & & \bigwedge^2 P_1 \\ & R \arrow[ur, "a_2"'] & \end{tikzcd}$$

commute. Thus it suffices to prove that the diagram

$$\begin{tikzcd}
\bigwedge^2 P_2 \arrow[rr, "\Lambda^2 f_2"] \arrow[dr, "a_3^*"'] & & \bigwedge^2 P_1 \\
& R \xrightarrow{a_3} \bigwedge^{r_3} P_2 = \bigwedge^2 P_2^* \arrow[ur, "\Lambda^2 b_2"'] &
\end{tikzcd}$$

commutes. Since we know that $f_2 = b_2 a_3'$, we have $\Lambda^2 f_2 = \Lambda^2 b_2 \, \Lambda^2 a_3'$. Thus it suffices to show that the diagram

$$\begin{tikzcd}
\bigwedge^2 P_2 \arrow[rr, "\Lambda^2 a_3'"] \arrow[d, equal] & & \bigwedge^2 P_2^* \arrow[d, equal] \\
\bigwedge^{r_3} P_2^* \arrow[r, "a_3^*"] & R \arrow[r, "a_3"] & \bigwedge^{r_3} P_2
\end{tikzcd} \tag{7.7}$$

commutes. We therefore prove the following.

LEMMA 7.3. *Let* (7.3) *be an exact sequence of oriented free modules satisfying condition* (7.3a) *and let* $a_3: R \to \Lambda^{r_3} P_2$, $a_3': P_2 \to P_2^*$ *be the maps constructed for this complex as in Theorems* 3.1 *and* 6.1. *Then the diagram* (7.7) *is commutative.*

Proof. To prove the lemma, we apply Theorem 1.2 to the sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_3 \xrightarrow{f_3} P_2 \xrightarrow{a_3'} P_2^*, \tag{7.8}$$

which was proved exact during the proof of Theorem 6.1. By Theorem 1.2, $\operatorname{rank}(a_3') = \operatorname{rank} P_2 - r_3$, and we are given that $\operatorname{rank} P_2 - r_3 = 2$. Thus by Theorem 3.1, there exists a unique map

$$c: R \to \bigwedge^2 P_2^*$$

making the diagram

$$\begin{tikzcd}
\bigwedge^2 P_2 \arrow[rr, "\Lambda^2 a_3'"] \arrow[dr, "a_3^*"'] & & \bigwedge^2 P_2^* \\
& R \arrow[ur, "c"'] &
\end{tikzcd} \tag{7.9}$$

commutative. We must show that $c = a_3$.

From the results of Section 3 applied to the exact sequence (7.8), we know that

$$0 \to R \xrightarrow{a_3} \bigwedge^{r_3} P_2 \xrightarrow{d} \bigwedge^{r_3+1} P_2 \otimes P_3^*$$

is exact, where $d = d_0^{f_3}$ in the notation of Lemma 3.2, and that the composition

$$\bigwedge^2 P_2 \xrightarrow{\wedge^2 a_3'} \bigwedge^2 P_2^* = \bigwedge^{r_3} P_2 \xrightarrow{d} \bigwedge^{r_3+1} P_2 \otimes P_3^*$$

is zero. Thus, by virtue of (7.9), the ideal $I(a_3) = \operatorname{Im}(a_3^*)$ annihilates the image of the composition:

$$R \xrightarrow{c} \bigwedge^2 P_2^* = \bigwedge^{r_3} P_2 \xrightarrow{d} \bigwedge^{r_3+1} P_2 \otimes P_3^*. \tag{7.10}$$

Since, by Theorem 3.1, $I(a_3)$ contains a nonzero divisor, the composition (7.10) is zero. Hence there exists a map $u\colon R \to R$ making the following diagram commutative:

$$\begin{tikzcd} R \arrow[rr, "a_3"] \arrow[dr, "u"'] & & \bigwedge^2 P_2^* \\ & R \arrow[ur, "c"'] & \end{tikzcd} \tag{7.11}$$

We will prove that $c = a_3$ by showing that $u = 1$. To do this, we will make the relationship between a_3 and a_3' more explicit.

We suppose that P_2 is free on a basis $\{p_i\}$ with dual basis $\{\rho_i\}$. The map a_3' was defined, in Theorem 6.1, by the commutativity of the following diagram

$$\begin{tikzcd} P_2 \arrow[rr, "a_3'"] \arrow[dr, "a_3 \otimes 1"'] & & P_2^* \\ & \bigwedge^2 P_2 \otimes P_2 \arrow[ur, "n"'] & \end{tikzcd}$$

Thus if we write $a_3(1) = \sum_{i<j} a_3(i,j)\, \rho_i \wedge \rho_j$, we have $a_3'(p_i) = \sum_{j \neq i} \pm\, a_3(i,j)\, \rho_j$, where the sign on the right is $+$ if and only if $i < j$.

Thus of we set, for $i < j$,

$$a_3(j, i) = -a_3(i, j),$$

We see that the (i, j)-th entry of the matrix of a_3' is

$$(a_3')_{ij} = \begin{cases} a_3(i,j) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

In particular, a_3' is alternating so the principal 2×2 minor involving rows i, j and columns i, j with $i < j$ is $(a_3(i, j))^2$. Thus we see that $(a_3(i, j))^2$ is the coefficient of $\rho_i \wedge \rho_j$ in the element $(\Lambda^2 a_3')(p_i \wedge p_j)$.

From (7.9) and (7.11) we have

$$\bigwedge^2 a_3' = a_3 \circ u \circ a_3^*.$$

Now $a_3 \circ u \circ a_3^*(p_i \wedge p_j) = u \sum a_3(i, j)\, a_3(k, l)\, \rho_k \wedge \rho_l$. Equating the coefficient of $\rho_i \wedge \rho_j$ in $(\Lambda^2 a_3')(\rho_i \wedge \rho_j)$ with the corresponding coefficient in $a_3 \circ u \circ a_3^*(p_i \wedge p_j)$, we get

$$(a_3(i, j))^2 = u(a_3(i, j))^2.$$

Since this holds for every i and j, we see that $(1 - u)[I(a_3)]^N = 0$ for N sufficiently large. Since, by Theorem 3.1, $I(a_3)$ contains a nonzero divisor, this implies that $u = 1$ as required. This establishes Lemma 7.3; we have $f_1' = a_2^*$ and consequently depth $(\mathrm{Im}(f_1')) \geqslant 2$.

To complete the proof of Theorem 7.1, we must now prove the exactness of (7.6) under the assumptions that a is a nonzero divisor and that depth$(\mathrm{Im}(f')) \geqslant 2$.

Since $f_2 = b_2 a_3'$ and $a_3' f_3 = 0$ (by (6.6)), we have $f_2 f_3 = 0$. By the argument following (6.5), we know that

$$P_3 \xrightarrow{f_3} P_2 \xrightarrow{a_3'} P_2^*$$

is exact so that rank $(a_3') = $ rank $P_2 - r_3$. But we are given that rank $P_2 - r_3 = 2$, so rank $(a_3') = 2$. Therefore rank $(f_2) \leqslant 2$. We will show that rank $(f_2) = 2$ and that depth $I(f_2) \geqslant 2$. By Theorem 1.2 applied to the sequence $0 \to P_n \to \cdots \to P_3 \to^{f_3} P_2 \to^{f_2} P_1$, this will establish the exactness of

$$P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1.$$

By Lemma 7.2 we know that $\wedge^2 a_3' = a_3 a_3^*$. Since $f_2 = ba_3'$, we have $\wedge^2 f_2 = (\wedge^2 b)(\wedge^2 a_3') = (\wedge^2 b)\, a_3 a_3^*$. Hence $I_2(f_2) = I(a_3^*)\, I((\wedge^2 b)\, a_3) = I(a_3^*)[\mathrm{Im}(a_3^* \wedge^2 b^*)]$. Since depth $I(a_3) \geqslant 3$ and, by hypothesis, depth $(\mathrm{Im}(a_3^* \wedge^2 b^*)) \geqslant 2$, we see that depth $(I_2(f_2)) \geqslant 2$. Since $\mathrm{rank}(f_2) \leqslant 2$, it follows that rank $(f_2) = 2$, $I(f_2) = I_2(f_2)$, and depth $(I(f_2)) \geqslant 2$. This proves the exactness of

$$P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 .$$

We now define a map $a_2 \colon R \to P_1$ by setting $a_2 = (\wedge^2 b)\, a_3$. Thus $\wedge^2 f_2 = a_2 a_3^*$. Since, essentially, $a_2^* = a_2'^*$, we know from (6.6) that the composition

$$P_2 \xrightarrow{f_2} P_1 = \bigwedge^2 P_1^* \xrightarrow{a_2^*} R$$

is zero. Since $f_1 = aa_2^*$, this proves that $f_1 f_2 = 0$. However, depth $I(a_2) = \mathrm{depth}(\mathrm{Im}(a_2^*)) \geqslant 2$ by hypothesis and, since $I(a_2) > 0$, $\mathrm{rank}(a_2) = 1$. Thus, because a is not a zero divisor, $\mathrm{rank}(f_1) = 1$ and depth $I(f_1) = \mathrm{depth}(aI(a_2)) \geqslant 1$. By Theorem 1.2 it follows that

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} R = P_0$$

is exact and the proof of Theorem 7.1 is complete.

Proof of Theorem 7.2. If we start with the exact sequence (7.1), we get the truncated sequence (7.3) and the triple $\{P_1, b_2, a_1\}$. Using this data, we construct the exact sequence (7.6) and we want to show that (7.6) spliced with (7.3) gives (7.1). But we know by Theorem 6.1 that our original map f_2 is the composition $b_2 a_3'$ (as illustrated in (7.2)). Also, from (7.2) we know that our original map f_1 is $a_1 a_2^*$. We have just shown that $f_1' = a_2^*$, so that $f_1 = a_1 f_1'$. This is exactly the map that should correspond to the data $\{P_1, b_2, a\}$ under the correspondence. Thus we have shown that starting with the exact sequence (7.1), and applying the two correspondences we return to the same sequence. That the composition of the correspondences in the reverse order is the identity also follows from Theorem 3.1 and 6.1.

Remark. It seems likely that the conditions given on the sequences (7.3) are sufficient to guarantee the existence of triples $\{P_1, b, a\}$ satisfying the conditions of Theorem (7.1). See Proposition 8.1.

8. Three-Generator Ideals of Homological Dimension 2

We now specialize to the case of homological dimension 2. In this case, the sequence (7.2) reduces to

$$0 \longrightarrow P_3 \xrightarrow{f_3} P_2 \tag{8.1}$$

with (a) $\operatorname{rank}(f_3) = \operatorname{rank} P_3 = \operatorname{rank} P_2 - 2$, and (b) $\operatorname{depth} I(f_3) \geqslant 3$. (It is known that for such a map f_3, we must have the equality: depth $I(f_3) = 3$. (See Remark 3 following Theorem 1.2.)

Furthermore, in this low-dimensional case, we have

$$a_3 = \bigwedge^{r_3} f_3 : \bigwedge^{r_3} P_3 = R \to \bigwedge^{r_3} P_2 = \bigwedge^{2} P_2{}^*.$$

Thus we see by Theorem 7.1, that an ideal I with three generators such that $hd_R R/I = 3$ is determined by the following data:

(i) a map f_3 satisfying conditions (a) and (b) above;

(ii) a map b: $P_2{}^* \to R^3$ such that depth $[\operatorname{Im}((\wedge^{r_3} f_3{}^*)(\wedge^2 b^*))] \geqslant 2$;

(iii) a nonzero divisor a.

We will show that, given a map f_3: $P_3 \to P_2$ satisfying condition (i), where P_3 and P_2 are free, there always exists a map b satisfying condition (ii).

Theorem 8.1. *Let R be a noetherian ring and let $f: R^n \to R^{n+2}$ be a map of free R-modules with* $\operatorname{rank}(f) = n$ *and depth $I(f) \geqslant 3$. Then there exists a map b: $R^{n+2*} \to R^3$ such that the image of $f_1 = (\wedge^n f^*)(\wedge^2 b^*)$: $R^3 \to R$ has depth 2. In particular, f is the second syzygy of a 3-generator ideal.*

We will show, in fact, that after a change of basis in R^{n+2}, the map b may be taken to be the projection onto the last three basis elements. This means that we can describe f_1: $R^3 \to R$ as the map whose coordinates are the three $n \times n$ minors of the matrix of f which contain the first $n - 1$ rows of the matrix.

For the proof, we will need a lemma. We are grateful to E. Graham Evans for his help in proving it.

Recall first that an elementary transformation is a product of transvections, that is, of square matrices with 1's on the diagonal and only one nonzero entry off the diagonal.

LEMMA 8.2. *Suppose that R is a noetherian ring, and that $a_1, \ldots, a_m \in R$ generate an ideal I of depth k. Then there is a set of generators $a_1', \ldots, a_m'$ of I, obtained from $a_1, \ldots, a_m$ by an elementary transformation, such that any subset of $\{a_1', \ldots, a_m'\}$ containing k elements is an R-sequence in some order.*

Proof of 8.2. We will prove by induction on l that for $l \leqslant k$, there is a sequence $a_1', \ldots, a_m'$ obtained from $a_1, \ldots, a_m$ by an elementary transformation, such that every subset of $\{a_1', \ldots, a_m'\}$ containing l elements is an R-sequence in some order.

The statement being vacuously satisfied for $l = 0$, we may suppose that every subset of $\{a_1, \ldots, a_m\}$ containing $l - 1$ elements is an R-sequence in some order. We may also suppose that any set of l elements of $\{a_1, \ldots, a_m\}$ which contains any of the elements $a_1, \ldots, a_j$ (and we allow j to be zero), is an R-sequence in some order.

Let $\{P_t\}_{t=1,\ldots,T}$ be the finite set consisting of every prime ideal which is associated to one of the ideals generated by $l - 1$ elements of $\{a_1, \ldots, a_m\}$. Since, by induction, an ideal generated by $l - 1$ elements of $\{a_1, \ldots, a_m\}$ is of depth $l - 1$, such ideals are depth-unmixed; that is, each of the prime ideals P_t has depth $l - 1$ [19, Theorem 130]. Since $l - 1 < k$, the ideal generated by $a_1, \ldots, a_m$ is not contained in the set $\bigcup_{t=1}^{T} P_t$. By [19, Theorem 124], this implies that there exist elements $r_i \in R$ such that

$$a_{j+1}' = a_{j+1} + \sum_{i \neq j+1} r_i a_i \notin \bigcup_{i=1}^{T} P_t .$$

The passage from $a = \{a_1, \ldots, a_m\}$ to

$$a' = \{a_1, \ldots, a_j, a_{j+1}', a_{j+2}, \ldots, a_m\}$$

is clearly elementary, and it is easy to see that any subset of l elements of a' which contains a_{j+1}' generates an ideal whose depth is l. On the other hand, no element other than a_{j+1} has been changed, so any subset of l elements of $\{a_1, \ldots, a_{j+1}', \ldots, a_m\}$ which contains any of the elements $a_1, \ldots, a_j, a_{j+1}'$ is an R-sequence in some order. Induction now completes the proof.

COROLLARY 8.3. *Let $g: R^n \to R^{n+1}$ be an $(n + 1) \times n$ matrix whose $n \times n$ minors generate an ideal of depth $\geqslant 2$. Then there exists an elementary transformation $e_0: R^{n+1} \to R^{n+1}$ such that every pair of $n \times n$ minors of $e_0 g$ generates an ideal of depth 2.*

Proof. We regard R^{n+1} as a free module with a chosen basis, which orients R^{n+1}. The map $\wedge^n g^*\colon \wedge^n R^{n+1*} \to R$ takes the elements of the canonical basis of $\wedge^n R^{n+1*}$ to the $n \times n$ minors of g. By Lemma 8.2, there is an elementary transformation $e\colon \wedge^n R^{n+1*} \to \wedge^n R^{n+1*}$ such that every pair of images of basis vectors under the map $(\wedge^n g^*)e$ generates an ideal of depth 2. Because e is an elementary $(n+1) \times (n+1)$ matrix, we have $e^{-1} = \wedge^n e^*$. Thus $(\wedge^n g^*)e = \wedge^n (g^* e^{*-1})$ and e^{-1} is the desired e_0 of the corollary.

The obvious generalization of Corollary 8.3 to $n \times m$ matrices with $n > m + 1$ is false, since if g is any $m \times n$ matrix, then the $m \times m$ minors of g obtained from a given $(m + 1) \times m$ submatrix generate an ideal whose depth is $\leqslant 2$ (Remark 3) following Theorem 1.2).

However, there is one true result in this direction, which we will use in the proof of Proposition 8.1.

THEOREM 8.4 [5]. *Let R be a local ring, and let g be an $m \times n$ matrix with entries in the maximal ideal of R. Suppose that $m \leqslant n$, $rk\, g = m$, and that depth $I(g) = n - m + 1$ (the largest possible value). Then, for all k with $m \leqslant k \leqslant n$, every $m \times k$ submatrix g' of g satisfies depth $(I(g')) = k - m + 1$.*

Remark. It would be interesting to know whether the following generalization of Lemma 8.2 holds: Let R be a local ring, and let g be an $m \times n$ matrix of rank m with entries in the maximal ideal of R. Suppose that depth $(I(g)) = k$ (It follows that $k \leqslant n - m + 1$.) Does there exist an $n \times n$ elementary transformation e such that every $m \times (m + k - 1)$ submatrix of ge has depth k?

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. The second statement follows from the first by Theorem 7.1.

To prove the first statement, we first choose bases for R^n and R^{n+2}, so that we can identify f with an $(n + 2) \times n$ matrix. Since depth is a local property, we may suppose that R is local. If the entries of f are not all in the maximal ideal of R, we can write

$$f = 1 \oplus f' : R^p \oplus R^{n-p} \to R^p \oplus R^{(n-p)+2}$$

for some $p > 0$, where the entries of the matrix of f' are all in the maximal ideal, and $I(f') = I(f)$. Thus we may suppose that all the entries of f are in the maximal ideal.

By Theorem 8.4 the $(n+1) \times n$ submatrix g of f consisting of the first $n+1$ rows of f satisfies depth $I(g) = 2$. After making an elementary transformation on the first $n+1$ rows of f if necessary, we may by Corollary 8.3 suppose that every pair of $n \times n$ minors of g generates an ideal of depth 2. Thus in particular, the 3 $n \times n$ minors of f which contain the first $n-1$ rows of f generate an ideal of depth 2.

Now let b: $R^{n+2*} \to R^3$ be the projection onto the last 3 coordinates. Identifying $\wedge^2 R^{3*} = R^3$ and $\wedge^2 R^{n+2} = \wedge^n R^{n+2*}$ as usual, we see that the image of $(\wedge^n f^*) \circ (\wedge^2 b^*)$: $R^3 \to R$ is the ideal generated by the 3 $n \times n$ minors of f which contain the first $n-1$ rows. Thus Theorem 8.1 is established.

9. Further Applications. Some Remarks on the Lifting Problem

Let S be a regular local ring, and let x be an element of S such that $R = S/(x)$ is also a regular local ring. Grothendieck's Lifting Problem is the following.

Given a finitely generated R-module M, does there exist a finitely generated S-module $M^{\#}$ such that x is a nonzero divisor on $M^{\#}$ and $M^{\#}/xM^{\#} \approx M$?

A module $M^{\#}$ with these properties is called a *lifting* of M. As is well-known, a positive solution to the lifting problem would yield a proof of Serre's multiplicity conjecture [7].

Laudal and Kleppe have recently shown that there are unliftable modules: in fact, they have shown that the graded coordinate ring of Serre's example [24] of a variety defined in characteristic $p \neq 0$ which cannot be lifted to characteristic 0, regarded as a module over a polynomial ring in characteristic p, is unliftable. However, the lifting problem in characteristic 0 is still open.

The lifting problem is connected with free resolutions via the following simple lemma.

Lemma 9.1 [7]. *Let S, R, and M be as above, and suppose that*

$$A: \quad F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

is exact, where the F_i are finitely generated free R-modules, and $\operatorname{Cok} f_1 = M$. *Suppose that*

$$B: \quad G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0$$

is a sequence of maps of free S-modules such that $g_1 g_2 = 0$ *and* $B \otimes_S R \approx A$. *Then* $M^{\#} = \operatorname{Cok} g_1$ *satisfies the conditions of the lifting problem.*

Proof. The condition that $B \otimes_S R \approx A$ implies the existence and exactness of the sequence of complexes:

$$0 \to B \xrightarrow{x} B \to A \to 0.$$

The associated exact homology sequence gives

$$\cdots \to H_1(A) \to M^{\#} \xrightarrow{x} M^{\#} \to M \to 0.$$

Since A is exact at F_1, $H_1(A) = 0$ so the sequence

$$0 \to M^{\#} \xrightarrow{x} M^{\#} \to M \to 0$$

is exact. It follows that $M^{\#}$ is a lifting of M.

Our work on the structure of free resolutions was originally motivated by the desire to lift modules, using Lemma 9.1. (This, for example, was the point of Theorem 3.4 in [6].)

Theorem 0 of this paper gives information which is sufficient to "lift" cyclic modules of homological dimension 2. (Lifting modules of homological dimension 1 is easy, using 9.1, as is the lifting of various other classes of modules (See [7]).

COROLLARY 9.2. *In the situation of the lifting problem, every cyclic R-module of homological dimension* 2 *is liftable.*

Proof. Let R/I be the module. Then R/I has a minimal free resolution of the form

$$0 \to R^{n-1} \xrightarrow{f_2} R^n \xrightarrow{f_1} R \to R/I \to 0.$$

We will produce a complex of free S-modules

$$B\colon \quad G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0$$

such that $B \otimes_S R \approx A$ where A is the complex

$$A\colon \quad R^{n-1} \xrightarrow{f_2} R^n \xrightarrow{f_1} R.$$

By Theorem 3.1 there exists a nonzero divisor $a = a_1 \in R$ such that

$$\begin{tikzcd} R^n \arrow[rr, "f_1"] \arrow[d] & & R \arrow[d, "a"] \\ \bigwedge^{n-1} R^{n*} \arrow[r, "\bigwedge^{n-1} f_2"] & R \approx & \bigwedge^{n-1} R^{n-1*} \end{tikzcd}$$

commutes. If we lift the maps f_2 and a to maps g_2: $S^{n-1} \to S^n$ and $a^{\#}$: $S \to S$ (for instance by lifting their matrices an element at a time), we can define the map g_1: $S^n \to S$ by the diagram

$$\begin{tikzcd} S^n \arrow[rr, "g_1"] \arrow[d] & & S \arrow[d, "a^{\#}"] \\ \bigwedge^{n-1} S^{n*} \arrow[r, "\bigwedge^{n-1} g_2{}^*"] & S \approx & \bigwedge^{n-1} S^{n-1*} \end{tikzcd}$$

The sequence

$$B\colon\quad S^{n-1} \xrightarrow{g_2} S^n \xrightarrow{g_1} S$$

has the property that $g_1 g_2 = 0$ and $B \otimes_S R \approx A$. Thus, by Lemma 9.1, Cok g_1 is a lifting of R/I.

Theorem 8.1 enables us to lift cyclic modules R/I of homological dimension 3, provided I is generated by three elements.

COROLLARY 9.3. *In the situation of the lifting problem, every cyclic R-module R/I, such that $hd(R/I) = 3$ and I is generated by* 3 *elements, is liftable.*

Proof. The resolution of R/I has the form:

$$A\colon\quad 0 \to R^n \xrightarrow{f_3} R^{n+2} \xrightarrow{f_2} R^3 \xrightarrow{f_1} R$$

and, by Theorem 8.1 A can be reconstructed from the map f_3 and the data $\{R^3, b_2, a_1\}$. If we lift f_3, b_2, and a_1 to maps

$$g_3\colon S^n \to S^{n+2}, \qquad b_2{}^{\#}\colon S^{n+2} \to S^{3*}, \qquad a_1{}^{\#}\colon S \to S,$$

then the map g_3 and the data $\{S^3, b_2{}^{\#}, a_1{}^{\#}\}$ clearly satisfy the conditions of Theorem 8.1. Thus we may construct from them maps g_2 and g_1 so that the sequence

$$B\colon\quad S^n \xrightarrow{g_3} S^{n+2} \xrightarrow{g_2} S^3 \xrightarrow{g_1} S$$

will reduce to A modulo x. By Lemma 9.1, B is exact and $\operatorname{Cok} g_1$ is a lifting of R/I.

Remark. In [7] it was incorrectly stated that Theorems 3.1 and 6.1 implied the liftability of all cyclic modules R/I of finite homological dimension, provided only that I is generated by three elements. In order to be able to lift all such modules, we see by Theorem 7.1 that it would be enough to lift an alternating matrix a_3' of rank 2 to an alternating matrix of rank 2. For then, it is easy to show that by lifting the matrices a_3', b_2, and a_1 associated with R/I, and defining a map g_1 in terms of a_3', b_2, and a_1 as above, we would get the result that $\operatorname{Cok} g_1$ is a lifting of R/I, even in the higher-dimensional case.

10. A Sketch of Results on Minors of Still Lower Order

Throughout this section, R will be a noetherian ring and

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 . \tag{10.1}$$

will be an exact sequence of free R-modules. We set $\operatorname{rank}(f_k) = r_k$.

The results of Sections 1, 3, and 6 give us reasonably strong information about the $r_k \times r_k$ minors of each f_k, with $k < n$, and about the minors of order $r_k - 1$ of each f_k for $2 \leqslant k \leqslant n$. It should be noted that knowing these two sets of minors specifies the resolution uniquely, at least in the case of a resolution of a cyclic module. This is because for $k > 2$, our knowledge of the minors of f_k of order $r_k - 1$ allows us to form the matrix of cofactors B of any minor A of f_k of order r_k. If we form the matrix of cofactors of B, we find ourselves with a copy of the original submatrix A, multiplied by $(\det A)^{r_k - 1}$. Since we have control over this determinant, and since it is a nonzero divisor "often enough," f_k is uniquely determined.

Unfortunately, the determination of the map f_k above involves division, so we have neither been able to use this information for the lifting problem, nor for a theorem on the existence of a sequence (10.1) with prescribed invariants a_k and b_k. Thus we have tried to examine the lower order minors of a resolution in the same way that succeeded in the cases of Theorems 3.1 and 6.1. The program is clear from the proofs of those theorems and runs as follows.

Define, for each $j \leqslant r_{k-1}$, a map $a_k^{(j)}\colon \Lambda^j P_{k-1} \to \Lambda^{j+r_k} P_{k-1}$ by the diagram

$$\begin{tikzcd}
\bigwedge^{j} P_{k-1} \arrow[rr, "a_k^{(j)}"] \arrow[dr, "a_k \otimes 1"'] & & \bigwedge^{j+r_k} P_{k-1} \\
& \bigwedge^{r_k} P_{k-1} \otimes \bigwedge^{j} P_{k-1} \arrow[ur, "m"'] &
\end{tikzcd}$$

Precisely as in the proofs of Theorems 3.1 and 6.1 it follows that, in the notation of Lemma 3.2, the composition of the maps in the following sequences

$$\bigwedge^{j} P_{k-1} \xrightarrow{a_k^{(j)}} \bigwedge^{j+r_k} P_{k-1} \xrightarrow{d_j^{f^k}} \bigwedge^{j+r_k+1} P_{k-1} \otimes P_k^* \tag{10.2}$$

and

$$\bigwedge^{r_{k-1}-j} P_{k-2}^* \xrightarrow{\Lambda^{r_{k-1}-j} f_{k-1}^*} \bigwedge^{r_{k-1}-j} P_{k-1}^* = \bigwedge^{j+r_k} P_{k-1} \xrightarrow{d_j^{f_k}} \bigwedge^{j+r_k+1} P_{k-1} \otimes P_k^*$$

are zero. If (10.2) were exact, there would exist a map, whose dual we shall denote by $c_k^{(j)*}$, from $\Lambda^{r_{k-1}-j} P_{k-2}^*$ to $\Lambda^j P_{k-1}$, making the following diagram commutative:

$$\begin{tikzcd}
\bigwedge^{r_{k-1}-j} P_{k-1} \arrow[rr, "\Lambda^{r_{k-1}-j} f_{k-1}"] \arrow[dr, "a_k^{(j)*}"'] & & \bigwedge^{r_{k-1}-j} P_{k-2} \\
& \bigwedge^{j} P_{k-1}^* \arrow[ur, "c_{k-1}^{(j)}"'] &
\end{tikzcd}$$

In this way, we would be able to express the minors of f_{k-1} of order $r_{k-1} - j$ in terms of $a_k^{(j)}$ and thus in terms of a_k. In particular, we would have $I_{r_{k-1}-j}(f_{k-1}) \subseteq I(a_k)$.

It turns out that (10.2) is exact—some of the time. The following statement, Conjecture 10.1, is a description of what we suspect the most general case to be. We have proved the conjecture in several special cases; we will describe a typical example.

CONJECTURE 10.1. *With the above notation and hypotheses, the sequence* (10.2) *is exact if*

$$(j-1)n \leqslant j(k-1) - 2. \tag{10.4}$$

Therefore, if (10.4) *holds, there exists a map* $c_k^{(j)}$ *making the diagram* (10.3) *commute.*

Remark 1. Theorem 3.1 is the special case $j = 0$. Theorem 6.1 is the special case $j = 1$.

Remark 2. From conjecture 10.1 it follows that if (10.4) is satisfied, then

$$I_{r_{k-1}-j}(f_{k-1}) \subset I(a_k). \tag{10.5}$$

We know no counterexample to (10.5) for which $j \leqslant k - 1$. If (10.5) always holds for $j \leqslant k - 1$, then Hackman's conjecture [17] that $r_k \geqslant k$ for $k < hd(\operatorname{Cok} f_1)$ is true. The special case of (10.5) given by Theorem 6.1 shows that $r_k \geqslant 2$ for $2 \leqslant k < hd(\operatorname{Cok} f_1)$ holds.

Remark 3. The peculiar inequality (10.4) that is required for the conjecture makes us suspect that this is not the right approach to the lower order minors of the maps of the sequence (10.1). Perhaps a better avenue would be to try to express lower order minors of f_{k-1} in terms of something having to do with lower order minors of f_k. So far, our attempts in this direction have failed.

We will now outline a program for proving the conjecture. The crucial point is the exactness of (10.2). Our method for attacking this question is to define a complex

$$0 \to Q_N \xrightarrow{g_N} Q_{N-1} \to \cdots \to Q_3 \xrightarrow{g_3} Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \tag{10.6}$$

such that the sequence $Q_2 \to^{g_2} Q_1 \to^{g_1} Q_0$ is (10.2), and to apply Theorem 1.2 to prove the exactness of (10.6).

The first step in this attack is to define Q_3 and g_3. In order to be able to apply Theorem 1.2, we must define g_3 in such a way that $g_2 g_3 = 0$ and $\operatorname{rank} g_3 + \operatorname{rank} g_2 = \operatorname{rank} Q_2$. By definition, $g_2 = a_k^{(j)}$. We know from Lemma 3.2 that $a_k^{(j)}$ maps into $(\operatorname{Ker} d_j^{f_k}) = (\wedge^{r_{k-1}-j} L)^*$, where $L = \operatorname{Cok} f_k$. It is easily seen that $a_k^{(j)}$ actually induces a map $a_k^{(j)}$: $\wedge^j L \to (\wedge^{r_{k-1}-j} L)^*$. If we localize at a nonzero divisor in $I(f_k)$, then L becomes free of rank r_{k-1}, and the map $a_k^{(0)}: R \to \wedge^{r_{k-1}} L^*$ induces an orientation on L^*. In this case, $a_k^{(j)}$ is the isomorphism $\wedge^j L \simeq \wedge^{r_{k-1}-j} L^*$ determined by this orientation.

With these facts in mind, we set $Q_3 = P_k \otimes \wedge^{j-1} P_{k-1}$, and let

$$Q_3 = P_k \otimes \bigwedge^{j-1} P_{k-1} \xrightarrow{g_3} \bigwedge^{j} P_{k-1}$$

be the map given by the $\wedge P_k$-module structure on $\wedge P_{k-1}$. It follows as in the proof of Lemma 3.2(d) that $\operatorname{Cok} g_3 = \wedge^j L$, and because $a_k^{(j)} = g_2$ factors through $\wedge^j L$, we have $g_2 g_3 = 0$. Moreover, we have seen that after inverting a nonzero divisor in $I(f_k)$, the sequence

$$Q_3 \xrightarrow{g_3} Q_2 \xrightarrow{g_2} Q_1$$

is exact. This shows that $\operatorname{rank} g_3 + \operatorname{rank} g_2 = \operatorname{rank} Q_2$, as required. We will now carry out this program for the case $k = n$.

We wish to resolve $\wedge^j L$, where, $L = \operatorname{Cok}(f_n)$ and f_n is a monomorphism. The following complex is exact by Theorem 1.2, and has the desired form

$$0 \to S_j(P_n) \otimes \bigwedge^0 P_{n-1} \xrightarrow{g_N} S_{j-1}(P_n) \otimes \bigwedge^1 P_{n-1} \to \cdots \to S_2(P_n) \otimes \bigwedge^{j-2} P_{n-1}$$

$$\xrightarrow{g_4} P_n \otimes \bigwedge^{j-1} P_{n-1} \xrightarrow{g_3} \bigwedge^{j} P_{n-1} \to Q_1 \to Q_0, \qquad (10.6')$$

where $S_t(P_n)$ denotes the t-th component of the symmetric algebra of P_n, and the differentials $g_{t+2}\colon S_t(P_n) \otimes \wedge^{j-t} P_{n-1} \to S_{t-1}(P_n) \otimes \wedge^{j-t+1} P_n$ are given as follows: since P_n is a finitely generated projective, the map $f_n\colon P_n \to P_{n-1}$ corresponds to an element $f^{\#} \in P_n{}^* \otimes P_{n-1}$. We may regard this element $f^{\#}$ as an element of bidegree $1 - 1$ in the bigraded algebra

$$S(P_n)^* \otimes \bigwedge P_{n-1},$$

where $S(P_n)^*$ denotes the graded dual of the symmetric algebra, in the sense of Bourbaki [4] (this algebra is somtimes called the algebra of divided powers). The natural map $P_n{}^* \otimes P_n \to R$ induces a unique algebra map $S(P_n)^* \otimes S(P_n) \to S(P_n)$, and this gives $S(P_n)$ the structure of an $S(P_n)^*$-module. Thus we may regard $S(P_n) \otimes \wedge P_{n-1}$ as an $(S(P_n)^* \otimes \wedge P_{n-1})$-module. We define the map

$$g_{t+2}\colon S_t(P_n) \otimes \bigwedge^{j-t} P_{n-1} \to S_{t-1}(P_n) \otimes \bigwedge^{j-t+1} P_{n-1}$$

to be multiplication by the element $f^{\#} \in S(P_n)^* \otimes \wedge P_{n-1}$.

The fact that (10.6′) is a complex follows from the fact that $(f^{\#})^2 = 0$.

It is easy to show, as in [9], that $\operatorname{Rad}(I(g_k)) = \operatorname{Rad}(I(f_n))$ for all k. By Theorem 1.2, the complex (10.6) is exact if and only if depth $(I(f_n)) \geqslant N = j + 2$ (the rank conditions are automatically satisfied by the complex (10.6′). Since depth $(I(f_n)) \geqslant n$, the complex (10.6′) is exact if $j \leqslant n - 2$. But this is precisely the inequality (10.4) when $k = n$, so we have established the conjecture for $k = n$.

Note that we get the additional information that the complex (10.2) is not exact if depth $(I(f_n)) = j + 1$. For if the complex (10.2) were exact, it would follow that the complex (10.6′) would be exact. Then by Theorem 1.2 it would follow that depth $(I(g_N)) \geqslant N = j + 2$. Since $\mathrm{Rad}(I(g_N)) = \mathrm{Rad}(I(f_n))$, we would get a contradiction.

11. A Homological Zoo

In this section we describe some examples of finite free resolutions, and remark on some of the problems they bring up. We strongly believe that further progress in the direction of this paper must depend, as this paper has, on a detailed study of examples.

We let R be a noetherian ring, and let

$$w, x, y, z \in R$$

be an R-sequence. As in earlier sections, we will deal with resolutions of cyclic modules rather than with resolutions of ideals.

We begin with two examples of resolutions of 3-generator ideals.

Example 1. A 3-generator ideal with 5 relations having homological dimension 2.

It is easily seen that the 3×3 minors of the matrix

$$f_3 = \begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ z & y & x \\ 0 & z & y \\ 0 & 0 & z \end{pmatrix} : R^3 \to R^5$$

generate the cube of the ideal (x, y, z); in particular, depth $I(f_3) = 3$. Therefore, by Theorem 7.1, we may find a 3-generator ideal of homological dimension 3 by choosing a map $b\colon R^{5*} \to R^3$ such that the image of the composite

$$R^3 = \bigwedge^2 R^{3*} \xrightarrow{\wedge^2 b^*} \bigwedge^2 R^5 = \bigwedge^3 R^{5*} \xrightarrow{\wedge^3 f_3^*} \bigwedge^3 R^{3*} = R$$

has depth 2. It turns out that if we choose b to be the projection onto the

1st, 4th, and 5th coordinates, the condition is satisfied. The resulting resolution is

$$0 \to R^3 \xrightarrow[\begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ z & y & x \\ 0 & z & y \\ 0 & 0 & z \end{pmatrix}]{f_3} R^5$$

$$\xrightarrow[\begin{pmatrix} 0 & -z^3 & yz^2 & -(y^2z - xz^2) & y^3 - 2xyz \\ y^2z - xz^2 & -xyz & x^2z & 0 & -x^3 \\ -(y^3 - 2xyz) & xy^2 - x^2z & -x^2y & x^3 & 0 \end{pmatrix}]{f_2} R^3$$

$$\xrightarrow[(x^3 \quad y^3 - 2xyz \quad y^2z - xz^2)]{} R$$

Examples of 3-generator ideals with any number of relations can be produced in a completely analogous way.

EXAMPLE 2. A 3-generator ideal with 5 relations having homological dimension 3.

It is well-known that 3-generator ideals can have any homological dimension [12; 16; 20]. Kaplansky observed that the "simplest" example of a 3-generator ideal with homological dimension 3 is the ideal $(wx, yz, wy + xz)$. Here is its resolution.

$$0 \to R \xrightarrow[\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}]{f_4} R^4 \xrightarrow[\begin{pmatrix} x & -w & 0 & 0 \\ 0 & 0 & -z & y \\ y & -z & -w & z \\ -z & 0 & 0 & w \\ 0 & y & -x & 0 \end{pmatrix}]{f_3} R^5$$

$$\xrightarrow[\begin{pmatrix} 0 & -(wy + xz) & yz & y^2 & z^2 \\ wy + xz & 0 & -wx & x^2 & w^2 \\ -yz & wx & 0 & -xy & -wz \end{pmatrix}]{f_2} R^3 \xrightarrow[(wx \quad yz \quad wy + xz)]{f_1} R.$$

Note that $I(a_3) \supseteq I_1(f_2)$ (as it must, by Theorem 6.1), but that $I(f_3) \not\supseteq I_1(f_2)$. The map f_3 is obtained from the dual of the Koszul relations on $f_4{}^*$, by composing with a projection to R^5. Free resolutions of 3-generator ideals of higher homological dimension could presumably be obtained in an analogous way, by "pruning" a Koszul complex suitably, and applying Theorem 7.1.

It seems reasonable, in the light of Theorem 8.1, and the theorems in [12, 16, and 20], to conjecture that "every" free resolution is the free resolution of a 3-generated ideal. More precisely, we have the following.

CONJECTURE. *Suppose that*

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_4 \xrightarrow{f_4} F_3 \xrightarrow{f_3} F_2 \tag{11.1}$$

is an exact sequence of free modules over a noetherian ring, and that $I(f_k) \geqslant k$ *for all* k. *Then* (11.1) *occurs as the left-hand part of a free resolution of a* 3*-generator ideal, in the sense that there exists*: (1) *a map* $c: F_2 \to R^{r_3+2} = F_2'$ *such that with* $f_3' = cf_3$, *we have* $\operatorname{rank} f_3' = r_3$ (*where* $r_3 = \operatorname{rank} f_3$), *and depth* $I(f_3') \geqslant 3$; *and* (2) *a map* $b: F_2'^* \to R^3$ *such that, if* a_3 *is the map defined by Theorem* 2.1 *for the sequence*

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_3 \xrightarrow{f_3'} F_2',$$

then depth $\operatorname{Im}(a_3^* \wedge^2 b^*) = 2$.

If these conditions were satisfied, Theorem 6.1 would yield maps

$$F_2' \xrightarrow{f_2} R^3$$

and

$$R^3 \xrightarrow{f_1} R$$

so that

$$0 \to F_n \to \cdots \xrightarrow{f_4} F_3 \xrightarrow{f_3'} F_2' \xrightarrow{f_2} R^3 \xrightarrow{f_1} R$$

is exact; in other words, a 3-generator ideal whose resolution agrees with (11.1) from the 3rd stage on.

We now return to homological dimension 2, and examine some 4-generator ideals.

EXAMPLE 3. A perfect 4-generator ideal of homological dimension 2. Recall that an ideal I is *perfect* if

$$hd(R/I) = ht \cdot (I).$$

If R is Macaulay, then it is well known that I is perfect if and only if I has finite homological dimension and R/I is Macaulay.

It is easily seen that the ideal (x, y^2, z^2, yz) is perfect. Here is its resolution:

$$0 \to R^2 \xrightarrow[\begin{pmatrix} z & 0 \\ 0 & y \\ y & z \\ x & 0 \\ 0 & x \end{pmatrix}]{f_3} R^5 \xrightarrow[\begin{pmatrix} y^2 & z^2 & yz & 0 & 0 \\ -x & 0 & 0 & z & 0 \\ 0 & -x & 0 & 0 & y \\ 0 & 0 & -x & -y & -z \end{pmatrix}]{f_2} R^4$$

$$\xrightarrow[(x \;\; y^2 \;\; z^2 \;\; yz)]{f_1} R.$$

It should be noticed here that the relations on the generators of the ideal, which are given by f_2, are "reduced Koszul relations"; that is, each of them may be obtained from one of the relations in the Koszul complex by dividing out the greatest common divisor of the terms in the relation. The same can be said of f_3. It turns out that the resolution for any ideal generated by monomials in an R-sequence has this form [25]. Note that in this example,

$$I(f_3) = I(a_3) \not\supseteq I_1(f_2),$$

(showing that Theorem 10.1 is best possible in this case), but that

$$I_1(f_3) \supseteq I_1(f_2).$$

This phenomenon points to one of the major sorts of open questions in this kind of analysis. We formulate a very special case explicitly:

Problem. In a free resolution of a perfect ideal of homological dimension 3, when is it true that

$$I_{r_3-1}(f_3) \supseteq I_{r_2-2}(f_2)\,?$$

When does

$$I_1(f_3) \supseteq I_1(f_2)\,?$$

For the resolution of perfect ideals of homological dimension 2, with 4 generators, we have been able to show that $\mathrm{Rad}(I_{r_3-1}(f_3)) \supseteq \mathrm{Rad}(I_1(f_2))$. To remove the radical sign remains a problem.

In the course of our work on free resolutions, we spent much time searching for an example of a perfect ideal of homological dimension 2 with 4 generators and 4 relations. It turns out that such things do not exist (more generally, a perfect ideal I of depth n such that $\mathrm{Ext}^n(R/I, R)$

is cyclic—that is, a Gorenstein ideal of depth n— cannot be minimally generated by precisely $n+1$ elements [9]. However, it can have $n+2$. (See Example 5, below.)

EXAMPLE 4. An imperfect ideal of homological dimension 2 with 4 generators and 4 relations:

The ideal is (x^2, y^2, xz, yz). Here is the resolution:

$$0 \to R \xrightarrow[\begin{pmatrix} z \\ -y^2 \\ x^2 \\ -xy \end{pmatrix}]{f_3} R^4 \xrightarrow[\begin{pmatrix} y^2 & z & 0 & 0 \\ -x^2 & 0 & z & 0 \\ 0 & -x & 0 & y^2 \\ 0 & 0 & -y & -x^2 \end{pmatrix}]{f_2} R^4 \xrightarrow[(x^2 \quad y^2 \quad xz \quad yz)]{f_1} R.$$

Note that here, $I(f_3) = I_1(f_3) \not\supseteq I_1(f_2)$.

EXAMPLE 5. A Gorenstein ideal of homological dimension 2 with 5 generators.

In a regular local ring R, a Gorenstein ideal may be defined as a perfect ideal for which the last module in the minimal free resolution of I is R. We have already remarked that if I is a Gorenstein ideal of depth n, then I cannot be minimally generated by $n+1$ elements [9]. The ideal whose resolution we will exhibit is $(x^3+y^3, x^3+z^3, xy, xz, yz)$:

$$R \xrightarrow[\begin{pmatrix} x^3+y^3 \\ x^3+z^3 \\ xy \\ xz \\ yz \end{pmatrix}]{f_3} R^5 \xrightarrow[\begin{pmatrix} 0 & 0 & z & 0 & -x \\ 0 & 0 & 0 & -y & x \\ -z & 0 & 0 & x^3 & y^3 \\ 0 & y & -x^3 & 0 & -z^3 \\ x & -x & -y^3 & z^3 & 0 \end{pmatrix}]{f_2} R^5 \xrightarrow[(x^3+y^3 \quad x^3+z^3 \quad xy \quad xz \quad yz)]{f_1} R.$$

Notice first that $f_3^* = f_1$. It is easy to show that the resolution of any Gorenstein ideal is isomorphic to its own dual. Note however, that something more is true here; we not only have $f_2 \simeq f_2^*$, but actually $f_2 = -f_2^*$. This is an example of the theorem about Gorenstein ideals of homological dimension 2 that we mentioned in the Introduction.

References

1. M. Auslander and D. A. Buchsbaum, Homological dimension in local rings, *Trans. Amer. Math. Soc.* **85** (1957), 390–405.
2. M. Auslander and D. A. Buchsbaum, Codimension and multiplicity, *Ann. of Math.* **68** (1958), 625–657.
3. M. Auslander and D. A. Buchsbaum, Unique factorization in regular local rings, *Proc. Nat. Acad. Sci.* **45** (1959), 733–734.
4. N. Bourbaki, "Éléments de Mathématiques, Algèbre," Chap. III, Hermann, Paris, 1970.
5. D. A. Buchsbaum, A generalized Koszul complex I, *Trans. Amer. Math. Soc.* **111** (1964), 183–197.
6. D. A. Buchsbaum, Complexes associated with the minors of a matrix, *Symposia Math.* **4** (1970), 255–283.
7. D. A. Buchsbaum and D. Eisenbud, Lifting Modules and a Theorem on Finite Free Resolutions, *in* "Ring Theory," pp. 63–74, Academic, New York, 1972.
8. D. A. Buchsbaum and D. Eisenbud, What makes a complex exact, *J. Algebra* **25** (1973), 259–268.
9. D. A. Buchsbaum and D. Eisenbud, Remarks on ideals and resolutions, *Symposia Math.* **XI** (1973), 193–204.
10. D. A. Buchsbaum and D. Eisenbud, Resolution algebras, Gorenstein ideals, and almost complete intersections. To appear.
11. L. Burch, On ideals of finite homological dimension in local rings, *Proc. Cam. Phil. Soc.* **64** (1968), 941–946.
12. L. Burch, A note on the homology of ideals generated by three elements in local rings, *Proc. Camb. Phil. Soc.* **64** (1968), 949–952.
13. J. Eagon and D. Northcott, Ideals defined by matrices, and a certain complex associated to them, *Proc. Royal Soc.* **a269** (1962), 188–204.
14. J. Eagon and D. Northcott, On the Buchsbaum–Eisenbud theory of finite free resolutions, *J. Reine Angew. Math.*
15. W. Gröbner, "Moderne Algebraische Geometrie," Springer, Wien, 1949.
16. T. Gulliksen, Tout ideal premier d'un anneau noethérien est associé à un ideal engendré par trois éléments, *C. R. Acad. Sci. Paris Ser. A* **271** (1970), 1206–1207.
17. P. Hackman, Exterior powers and homology, to appear.
18. D. Hilbert, Über die theorie der algebraischen formen, *Math. Ann.* **36** (1890), 473–534.
19. I. Kaplansky, "Commutative Rings," Allyn and Bacon, Boston, 1970.
20. P. Kohn, Ideals generated by 3 elements, *Proc. Amer. Math. Soc.* **35** (1972), 55–58.

21. R. E. MacRae, On an applications of fitting invariants, *J. Algebra* 2 (1965), 153–169.
22. C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, *Pub. I.H.E.S. Paris* **42**, 1973.
23. M. Schaps, Nonsingular deformations of space curves, using determinantal schemes, Thesis, Harvard, 1972.
24. J.-P. Serre, Examples de variété projectives en caractéristique p non relevables en caractéristique zéro, *Proc. Nat. Acad. Sci.* **47** (1961), 108–109.
25. D. Taylor, Ideals generated by monomials in an R-sequence, Thesis, University of Chicago, 1966.