On A PROBLEM IN LINEAR ALGEBRA

David A. Buchsbaum

and

David Eisenbud

In its vaguest and most tantalizing form, the problem referred to in the title of this paper is to say something about the solution of systems of linear equations over a polynomial ring $R = K[X_1,\dots,X_n]$, where K is a field. The difficulty is, of course, that a system of linear equations over R may not possess a set of linearly independent solutions from which all solutions may be obtained by forming linear combinations with coefficients in R .

Hilbert's Syzygy Theorem suggests a promising approach to this problem: Let φ_1 be a system of linear equations, which we will regard as a map of free R-modules, say $\varphi_1 \colon F_1 \longrightarrow F_o$. If we form a free resolution of the cokernel of φ_1 , that is an exact sequence of free R-module of the form

$$(1) \qquad F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} F_{n-2} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_o ,$$

then the Syzygy Theorem tells us that the system of linear equations represented by φ_{n-1} does have a full set of linearly independent solutions. More precisely, it tells us this in the graded or local cases; in general, Ker φ_{n-1} is merely stably free. Thus, replacing F_n by Ker φ_{n-1} in the sequence (1), we may replace (1) by a finite free (or stably free) resolution of the form

$$(2) \qquad 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_o .$$

In order to use the approach to linear equations that Hilbert's Theorem suggests, it seems to be necessary to study the relations that must hold between the maps φ_i in a finite free resolution of the form of (2). The first result, establishing a relationship between the maps φ_i , was also obtained by Hilbert [6]. Letting $\overset{n}{\Lambda} X$ denote the n^{th} exterior power of a module X, we state it as;

<u>Theorem 1.</u> Suppose that R is a noetherian ring. If

$$(3) \qquad 0 \longrightarrow R^n \xrightarrow{\varphi_2} R^{n+1} \xrightarrow{\varphi_1} R$$

is an exact sequence of free R-modules, then there exists a non-zerodivisor $a \in R$

such that, after making the canonical identification $\overset{n}{\wedge} R^{n*} \approx R$ and $\overset{n}{\wedge} R^{n+1*} \approx R^{n+1}$, we have $\varphi_1 = a\overset{n}{\wedge} \varphi_2{}^*$.

(Hilbert actually proved this only when R is the graded polynomial ring in two variables over a field; he did it to illustrate the application of the Hilbert function. The first proof that works in the generality in which we have stated the theorem is due to Burch [5]. The most elementary proof is contained in [7]. We shortly give a new proof of this result.)

Theorem 1 can be applied to a special case of Grothendieck's Theorem in a way that we will now describe. To do so, we will shift our point of view from polynomial rings to regular local rings; just as in the polynomial case, every module over a regular local ring has a finite free resolution.

Grothendieck's lifting problem is the following: let S be a regular local ring, and let $x \in S$ be such that $R = S/(x)$ is regular. Given a finitely generated R-module M , does there exist a finitely generated S-module $M^{\#}$ such that

(1) $M^{\#}/xM^{\#} \approx M$;

(2) x is a non-zerodivisor on $M^{\#}$?

If a module $M^{\#}$ satisfying (1) and (2) exists, it is called a lifting of M .

That the lifting problem is closely related to the structure of free resolutions is shown by the next lemma.

<u>Lemma 2.</u> Let S be any ring, and let $x \in S$ be a non-zerodivisor. Let $R = S/(x)$, and let

$$F: F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_o$$

be an exact sequence of free R-modules. Suppose that

$$F^{\#}: F_2^{\#} \xrightarrow{\varphi_2^{\#}} F_1^{\#} \xrightarrow{\varphi_1^{\#}} F_o$$

is a sequence of free S-module which reduces modules (x) to $\mathbb{F}$. If $\varphi_1^{\#} \varphi_2^{\#} = 0$, then Coker $(\varphi_1^{\#})$ is a lifting of $\mathrm{Coker}(\varphi_1)$.

<u>Proof</u>: The hypothesis allows us to construct a short exact sequence of complexes

$$0 \longrightarrow \mathbb{F}^{\#} \xrightarrow{x} \mathbb{F}^{\#} \longrightarrow \mathbb{F} \longrightarrow 0 \quad .$$

The associated exact sequence in homology contains

$$(4) \quad H_1(\mathbb{F}) \longrightarrow H_o(\mathbb{F}^{\#}) \xrightarrow{x} H_o(\mathbb{F}^{\#}) \longrightarrow H_o(\mathbb{F}) \longrightarrow 0 \quad .$$

On the other hand, $H_o(\mathbb{F}^{\#}) = \mathrm{Coker}(\varphi_1^{\#})$; $H_o(\mathbb{F}) = \mathrm{Coker}(\varphi_1)$ and, by hypothesis $H_1(\mathbb{F}) = 0$. Thus (4) becomes

$$0 \longrightarrow \mathrm{Coker}(\varphi_1^{\#}) \xrightarrow{x} \mathrm{Coker}(\varphi_1^{\#}) \longrightarrow \mathrm{Coker}(\varphi_1) \longrightarrow 0 ,$$

and the lemma is proved.

Lemma 2 implies the liftability of various classes of modules. For example, in the situation of the lifting problem, any module M of homological dimension one can be lifted: If $0 \longrightarrow F_1 \xrightarrow{\varphi_1} F_o$ is a free presentation of M , then there exists a homomorphism $\varphi_1^{\#}$ of free S-modules which reduces to φ_1 modulo x (for example, to construct $\varphi_1^{\#}$ one can choose bases of F_1 and F_2 and lift each element of the matrix of φ_1). By lemma 2, Coker $(\varphi_1^{\#})$ is a lifting of M .

Theorem 1 can be applied to the lifting problem through the use of lemma 2. Together they imply the liftability of cyclic modules of homological dimension 2. For, if (3) is the free resolution of such a module, we may choose a map $\varphi_2^{\#}$ of free S-modules and an element $a^{\#} \in S$ which reduce, modulo (x), to φ_2 and a respectively. Letting $\varphi_1^{\#} = a^{\#} \overset{n}{\Lambda} \varphi_2^{\#}$, it is easy to see that $\varphi_1^{\#} \varphi_2^{\#} = 0$. By theorem 1, $\varphi_1^{\#}$ reduces to φ_1 modulo (x) , so lemma 2 applies and shows that $\mathrm{Coker}(\varphi_1^{\#})$ is a lifting of $\mathrm{Coker}(\varphi_1)$.

We now return to the problem with which we began: what can one say about the relations between the maps φ_i in (2) ? If the F_i were vector spaces, the exactness of (2) would be equivalent to a condition on the ranks of the maps φ_i .

In the case of, say, a local ring R , what additional condition or conditions must be imposed in order to ensure exactness of a sequence of free R-modules? In order to answer this, we introduce some terminology and notation.

Let R be a commutative ring and $\varphi: F \longrightarrow G$ a map of finitely generated free R-modules. We define the <u>rank of φ</u> to be the largest integer r such that $^r\Lambda\varphi: {}^r\Lambda F \longrightarrow {}^r\Lambda G$ is not zero. If $r = \mathrm{rank}(\varphi)$, we denote by $I(\varphi)$ the ideal of R generated by the minors of φ of order r .

The definition of $I(\varphi)$ makes sense if one chooses bases of F and G and writes the matrix associated to φ . It can also be shown that the map $\varphi: F \longrightarrow G$ induces, for every k , a map $^k\Lambda G^* \otimes {}^k\Lambda F \xrightarrow{\Delta_k} R$ (where $G^* = \mathrm{Hom}_R(G,R)$) and $I(\varphi) = \mathrm{Im}(\Delta_r)$.

With this notation, we can state

Theorem 3 [2]. Let R be a noetherian commutative ring, and

$$\mathbb{F}: 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_o$$

a complex of free R-modules. Then $\mathbb{F}$ is an exact sequence if and only if

i) $\operatorname{rank}(\varphi_{k+1}) + \operatorname{rank}(\varphi_k) = \operatorname{rank}(F_k)$ for $k = 1,\ldots,n$ and

ii) $I(\varphi_k)$ contains an R-sequence of length k for $k = 1,\ldots,n$.

We make the convention that condition ii) is automatically satisfied if $I(\varphi_n) = R$. As a result, when R is a field, condition i) becomes the only (and the usual) condition for the exactness of a complex of vector spaces.

Theorem 3 provides one very quick proof of Theorem 1. For suppose

$\mathbb{F}: 0 \longrightarrow R^{n-1} \xrightarrow{(a_{ij})} R^n \xrightarrow{(y_i)} R \longrightarrow R/I \longrightarrow 0$ is exact. Letting $\varphi_2 = (a_{ij})$ and $\varphi_1 = (y_i)$, we know by Theorem 3 that $\operatorname{rank}(\varphi_2) = n-1$ and $I(\varphi_2)$ contains an R-sequence of length two. But $I(\varphi_2) = (\Delta_i)$ where $\{\Delta_i\}$ are the minors of order $n-1$ of the matrix (a_{ij}) . Thus, the sequence $0 \longrightarrow R \xrightarrow{(\Delta_i)} R^n \xrightarrow{\varphi_2^*} R^{n-1}$ is exact since the composition is clearly zero and Theorem 3 applies.

Because the sequence - $0 \longrightarrow R \xrightarrow{(y_i)} R^n \xrightarrow{\varphi_2^*} R^{n-1}$ is of order two, we get a map of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{(y_i)} & R^n & \xrightarrow{\varphi_2^*} & R^{n-1} \\ & & \downarrow a & & \big| & & \big| \\ 0 & \longrightarrow & R & \xrightarrow{(\Delta_i)} & R^n & \xrightarrow{\varphi_2^*} & R^{n-1} \end{array}$$

and thus $y_i = a\Delta_i$ for $i = 1,\ldots,n$.

Although Theorem 3 provides us with a nice proof of Theorem 1, it is not clear how that gets us any further into the problem of determining relations among the maps of an arbitrary finite free resolution. In [4], we have exploited Theorem 3 to obtain the following two results:

Theorem 4 [4, Th. 3-1]. Let R be a noetherian ring, let (2) be an exact sequence of free R-modules, and let $r_k = \operatorname{rank}(\varphi_k)$. Then for each k, $1 \leq k \leq n$, there exists an unique homomorphism $a_k : R \longrightarrow \Lambda^{r_k} F_{k-1} \approx \Lambda^{r_{k-1}} F^*_{k-1}$ such that

i) $a_n = \Lambda^{r_n} \varphi_n : R = \Lambda^{r_n} F_n \longrightarrow \Lambda^{r_n} F_{n-1}$

ii) for each $k < n$, the diagram

$$\begin{array}{ccc} \Lambda^{r_k} F_k & \xrightarrow{\Lambda^{r_k} \varphi_k} & \Lambda^{r_k} F_{k-1} \\ {\scriptstyle a^*_{k-1}} \searrow & & \swarrow {\scriptstyle a_k} \\ & R & \end{array}$$

commutes.

Using the maps a_k, we may define maps $a'_k : \Lambda^{r_{k-1}-1} F_{k-1} \longrightarrow F^*_{k-1}$ to be the composite:

$$\Lambda^{r_{k-1}-1} F_{k-1} = R \otimes \Lambda^{r_k+1} F^*_{k-1} \xrightarrow{a_k \otimes 1} \Lambda^{r_k} F_{k-1} \otimes \Lambda^{r_k+1} F^*_{k-1} \xrightarrow{n} F^*_{k-1}$$

where $n : \Lambda^{r_k} F_{k-1} \otimes \Lambda^{r_k+1} F^*_{k-1} \longrightarrow F^*_{k-1}$ is the usual action of ΛF_{k-1} on ΛF^*_{k-1} (see [1]).

Theorem 5. [4, Th. 6.1]. Let notation and hypothesis be as in Theorem 4. Then for $k \geq 2$, there exist maps $b_k : F^*_k \longrightarrow \Lambda^{r_{k-1}} F_{k-1}$ making the diagram

$$\begin{array}{ccc} \Lambda^{r_{k-1}} F_k & \xrightarrow{\Lambda^{r_k-1} \varphi_k} & \Lambda^{r_k-1} F_{k-1} \\ {\scriptstyle a'_{k-1}} \searrow & & \swarrow {\scriptstyle b_k} \\ & F^*_k & \end{array}$$

commute.

Theorems 4 and 5 give us fairly strong information about the relations of the minors of orders r_k and $r_k - 1$ of the maps φ_k in a finite free resolution (2). If we have the exact sequence

$$F;\ 0 \longrightarrow R^{m-2} \xrightarrow{\varphi_3} R^m \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R\ ,$$

then $r_1 = 2$, $r_1 - 1 = 1$, and it is possible to use Theorems 4 and 5 to express the maps φ_1 and φ_2 completely in terms of the minors of φ_3 of order m-2 (see [4]). Consequently just as Theorem 1 was applied to the lifting problem for cyclic modules of homological dimension 2, Theorems 4 and 5 may be applied to show that cyclic modules R/I of homological dimension 3 may be lifted, provided I is generated by 3 elements.

In the general case of a cyclic module R/I of homological dimension 3, our results applied to a resolution of R/I:

$$0 \longrightarrow R^p \xrightarrow{\varphi_3} R^m \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

only give us information about the minors of order n-1 and n-2 of φ_1 . One might therefore hope that further information about the lower order minors of φ_1 and φ_2 might be obtained from information about the lower order minors of φ_3 . This idea, of course, is completely demolished when one finds resolutions of the form

$$(5)\quad 0 \longrightarrow R \xrightarrow{\varphi_3} R^n \xrightarrow{\varphi_n} R^n \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0,$$

particularly if I contains an R-sequence of length three. If R is a regular local ring, this turns out to be the case precisely when R/I is a Gorenstein ring. It is possible to show, however, that the lifting problem for 5) in the Gorenstein case reduces to a problem of lifting

$$0 \longrightarrow R^p \xrightarrow{\psi_3} R^m \xrightarrow{\psi_2} R^4 \xrightarrow{\psi_1} R \longrightarrow R/J \longrightarrow 0$$

where J is an ideal generated by four elements and contains an R-sequence of length three. Since $p = m-4+1 = m-3$, the map ψ_3 will have lower order minors than those of order p, provided $m - 3 > 1$ i.e. $m > 4$. The question arises: is it possible to have an ideal I in a regular local ring R such that

i) R/I is a Gorenstein ring
ii) hd_R R/I = n and
iii) I is minimally generated by n + 1 elements?

In [3], we show that this situation cannot arise. In fact, the question arises as to what restrictions there are on the number of generators of an ideal I in a regular local ring R when R/I is Gorenstein. If $\dim R = 3$ and $\dim R/I = 0$, we have found such ideals I generated by five, seven, and nine elements, but none that are generated by an even number of elements.

This appears to be tied up with questions about the possible (or probable) skew-symmetry of the map φ_2 in the resolution of such an ideal:

$$0 \longrightarrow R \xrightarrow{\varphi_3} R^n \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

A computer program worked out by R. Zibman for the Brandeis PDP-10 computer has provided numerous examples of such ideals I and may yield some helpful information on the problem. In any event, it seems evident that our problem in linear algebra has many ramifications, the nature of which we are just beginning to discover.

Bibliography

1. N. Bourbaki, Elements de Mathematiques, Algebre, Ch. III, Hermann, 1970.
2. D.A. Buchsbaum and D. Eisenbud, What makes a complex exact, J. Alg. to appear.
3. _________ __________, Remarks on ideals and resolutions, Symposia Math. to appear.
4. _________ __________, Some structure theorems for finite free resolutions, To appear.
5. L. Burch, On ideals of finite homological dimension in local rings, Proc. Cam. Phil. Soc. 64 (1968) 941-946.
6. D. Hilbert, Uber die Theorie der algebraischen Formen, Math. Ann. (1890), 473-534.
7. I. Kaplansky, Commutative Rings, Allyn and Bacon, 1970.