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Every Algebraic Set in n-Space Is the Intersection of n Hypersurfaces

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In this paper we prove the theorem stated in the title for affine and projective n-space. More generally, we show that a radical ideal in an n-dimensional noetherian polynomial ring is always the radical of an ideal generated by n elements (Theorem 1), and we prove a corresponding theorem for graded rings (Theorem 2).

This strengthens the classical result (announced by Kronecker in 1882 [6] in the case of a polynomial ring in n variables over a field) that any radical ideal in an n-dimensional noetherian ring is the radical of an ideal generated by $n+1$ elements.

The history of these results is rather interesting. In 1891, 9 years after Kronecker had announced his theorem, Vahlen produced an example which, he claimed, showed that Kronecker's result was the best possible. The example he gave is a curve in complex projective 3-space which, he "showed", is not the intersection of 3 hypersurfaces [12]. Vahlen's error seems to have gone undetected until 1942, when Perron [7] exhibited 3 hypersurfaces whose intersection is the curve in question. (The year before, Van der Waerden [13] had given the first modern proof of Kronecker's theorem.) In 1961 Kneser [7] showed that the existence of Perron's hypersurfaces was not an accident by proving that every curve in 3-space is an intersection of 3 hypersurfaces. Our proofs of Theorems 1 and 2 rely on a modification (really an algebraization) of Kneser's idea.

It seems worthwhile to mention two problems that remain open in this area. Murthy [8] has recently shown that if K is a field, then in the ring $K[X, Y, Z]$, any ideal of height 2 which is locally a complete intersection can be generated by 3 elements (a somewhat weaker version of this is given by Abhyankar in [1]). It would be very interesting to have a higher-dimensional analogue of Murthy's result.

Murthy also gives an example to show that ideal corresponding even to a nonsingular curve in 3-space need not be generated by 2 elements (see also [11]). But the major question in the field remains open: Is every curve (even nonsingular) in 3-space the (set-theoretic) intersection of 2 hypersurfaces? The answer to the corresponding question in 4-space is

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negative. The surface S corresponding to the ideal $(x_1, x_2) \cap (x_3, x_4) \subseteq K[x_1 \ldots x_4]$ is not the intersection of 2 hypersurfaces. One way to see this is to use the theorem of Hartshorne [4] that any set-theoretic complete intersection is connected in codimension 2, whereas

$$S - \{(0,0,0,0)\}$$

is not connected.

In [3] we have conjectured a theorem which would include the result of this paper and yield a strengthening, for polynomial rings, of Serre's well-known theorem on free summands of projective modules.

The main result of this paper is stated and proved in Section 1. Section 2 contains the additional arguments necessary for the projective case.

1. The Affine Case

If I is an ideal of a ring R, we will write $\sqrt{I}$ for the *radical* of I; that is, $\sqrt{I}$ is the intersection of the prime ideals of R which contain I.

Theorem 1. *Let R be a noetherian ring of dimension n, and suppose that R is a polynomial ring of the form $R = S[x]$ for some ring S. Let $I \subseteq R$ be an ideal. Then there exist n elements $g_1, \ldots, g_n \in I$ such that*

$$\sqrt{I} = \sqrt{(g_1, \ldots, g_n)}.$$

Remark. By using the technique of basic elements as in [2] one can do slightly better. Under the hypotheses of Theorem 1, elements $g_1, \ldots, g_n \in I$ may be chosen to satisfy the conclusion of Theorem 1, and to have in addition the property that for any prime ideal P of R, if $P \supseteq I$ then $I_P \not\supseteq (g_1, \ldots, g_n)$.

Corollary 1. *Every algebraic set in affine n-space is the intersection of n hypersurfaces.*

Proof of Corollary 1. This is nothing but Theorem 1 applied to the ring $R = K[x_1, \ldots, x_n]$ with K a field. ||

Proof of Theorem 1. We will proceed by induction on n, using the fact that $\dim S = n - 1$.

Suppose first that $n = 1$, so that S is artinian. Let N be the nilpotent radical of S, and set

$$\bar{R} = R/NR = (S/N)[x]$$

$$\bar{I} = (I + NR)/NR \subseteq \bar{R}.$$

Since S/N is a finite direct product of fields, $\bar{R}$ is a principal ideal ring, so there is an element $g \in I$ whose image $\bar{g} \in \bar{I}$ generates $\bar{I}$. Since NR is nilpotent, it is contained in every prime ideal of R, so $\sqrt{(g)} = \sqrt{I}$ as desired.

Now suppose $n>1$. Let $P_1, \dots, P_k$ be the minimal primes of S, and let $U=S-\bigcup_{i=1}^{k} P_i$. Clearly, U is a multiplicatively closed set, and S_U is 0-dimensional. Thus $R_U=S_U[x]$ is 1-dimensional, so by the case $n=1$, there is an element $g_1 \in I_U$ such that $\sqrt{(g_1)_U}=\sqrt{I_U}$ in R_U. Multiplying g_1 by an element of U if necessary, we may assume that $g_1 \in I$. Since I is finitely generated, there is an element $u \in U$ such that

$$uI \subseteq \sqrt{(g_1)}. \tag{1}$$

Because

$$u \notin \bigcup_{i=1}^{k} P_i, \qquad \dim S/(u) \leqq n-2,$$

and the dimension of $R/(u)=(S/(u))[x]$ is $\leqq n-1$. Let $R^*=R/(u)$, and let $I^*=(I+(u))/(u)$. By the induction hypothesis, there exist elements $g_2^*, \dots, g_n^* \in I^*$ such that

$$\sqrt{I^*}=\sqrt{(g_2^*, \dots, g_n^*)} \tag{2}$$

in R^*. Let $g_2, \dots, g_n \in I$ be elements which reduce to $g_2^*, \dots, g_n^*$ modulo (g_1).

We assert that $\sqrt{I}=\sqrt{(g_1, \dots, g_n)}$. To prove this, we must show that if P is any prime ideal of R such that $P \supseteq (g_1, \dots, g_n)$, then $P \supseteq I$. Since $P \supseteq (g_1)$, it follows from Eq. (1) that $P \supseteq I$ or $P \supseteq (u)$. In the second case, the ideal

$$P^*=P/(u) \subseteq R^*$$

is a prime ideal, so by (2), $P^* \supseteq I^*$. Thus in this case $P=P+(u) \supseteq I+(u)$. In either case, $P \supseteq I$. ||

2. The Projective Case

In order to obtain the projective analogue of Corollary 1, we need a graded version of Theorem 1.

For us, a *graded ring* S will be positively graded, $S=\sum_{i \geqq 0} S^{(i)}$, and we set $S^+=\sum_{i>0} S^{(i)}$. We will always assume that S^+ is generated, as an ideal, by $S^{(1)}$. If $R=S[x]$, we regard R as a graded ring in the usual way:

$$R^{(i)}=\sum_{i=j+k} S^{(j)} x^k.$$

If R is any graded ring then a *relevant* prime ideal of R is a homogeneous prime ideal of R which does not contain R^+. If R is noetherian, the *projective dimension* of R is the length of a maximal chain of relevant prime ideals. Note that proj dim $R=-1$ is possible, and is true if and only if R^+ is nilpotent. If S is a noetherian graded ring of projective dimension, then proj dim $S[x] \geqq n+1$, but the inequality may be strict.

Theorem 2. *Let R be a graded noetherian ring, and suppose that R is a graded polynomial ring of the form $R=S[x]$ for some graded ring S of projective dimension $n-1$. Suppose that $I\subseteq S^+R$ is a homogeneous ideal. Then there exist n homogeneous elements $g_1,\dots,g_n\in I$ such that*

$$\sqrt{I}=\sqrt{(g_1,\dots,g_n)}.$$

Corollary 2. *Every algebraic set in projective n-space is the intersection of n hypersurfaces.*

Proof of Theorem 2. As in the proof of Theorem 1, we proceed by induction on n. Thus if $n=0$, S^+ is nilpotent, so $\sqrt{I}$ is the nilpotent radical of R, which as before is the radical of the ideal (0), generated by the empty set of elements.

Now suppose $n\geqq 1$, and let $P_1,\dots,P_k$ be the minimal relevant primes of S. We will prove that there exist elements $u\in S^+$ and $g_1\in I$ such that

$$uI\subseteq\sqrt{(g_1)} \tag{1'}$$

and $u\notin\bigcup_{i=1}^{k}P_i$. Once this is done, the proof may be completed by following the line of argument in the proof of Theorem 1, starting from Eq. (1).

We will need the following elementary fact about polynomials:

Lemma. *Let S be any ring and let $f,g\in S[x]$ be polynomials whose degrees in x are d and e respectively, with $d\leqq e$. If $u\in S$ is the leading coefficient of f, then for all $n>e-d$ there exist polynomials h and r so that*

$$u^n g=fh+r$$

and the degree of r in x is $<d$. Moreover, if S is graded and f and g are homogeneous, h and r may be chosen homogeneous as well.

Proof. All this follows immediately from the elementary method of dividing one polynomial by another. ‖

Returning to the proof of Theorem 2, we set, for $i=1,\dots,k$

$$I_i=(I+P_iR)/P_iR\subseteq R/P_iR.$$

For each i, let $h_i\in I$ be an element which reduces modulo P_iR to an element $h_i^*\in I_i$ having the lowest possible degree in x. For each i, choose $u_i\in S^+$ as follows: if $h_i^*=0$, then choose u_i to be any homogeneous element not contained in P_i. If $h_i^*\neq 0$, choose u_i to be a homogeneous element reducing mod P_i to the leading coefficient of h_i^*.

Now choose, for each $i=1,\dots,k$, a homogeneous element, $s_i\in S$ such that $s_i\in\left(\bigcup_{j=1}^{k}P_j\right)-P_i$, and choose $s\in S^{(1)}-\bigcup_{i=1}^{k}P_i$; this last choice can be made because if $S^{(1)} \subseteq \bigcup_{i=1}^{k} P_i$, then $S^+ \subseteq \left(\bigcup_{i=1}^{k} P_i\right) \cup \left(\sum_{i>1} S^{(i)}\right)$, which leads to the contradiction $S^+ \subseteq P_i$ for some i.

Multiplying each s_i, h_i, and u_i by a power of s, we may clearly assume that for all i and j,

$$\deg(s_i) = \deg(s_j)$$
$$\deg(h_i) = \deg(h_j)$$
$$\deg(u_i) = \deg(u_j).$$

Let $g_1 = \sum_{i=1}^{k} s_i h_i$, and let $u = \left(s\left(\sum_{i=1}^{k} s_i u_i\right)\right)^N$, with N yet to be determined. Clearly, $u \notin \bigcup_{i=1}^{k} P_i$.

We assert that (1′) is satisfied by this choice of g_1 and u if we take N to be sufficiently large. To see this, fix an index i, and let g_1^*, u^*, u_i^*, h_i^*, s_i^*, and s^* be the images of g_1, u, u_i, h_i, s_i, and s in $R/P_i R$. Then $g_1^* \in I_i$ is s_i^* times h_i^*, so the degree of g_1^* in x is minimal among the degrees in x of elements of I_i. Moreover, if N is large, then $u^* = (s^* s_i^* u_i^*)^N$ is a multiple of a large power of the leading coefficient of g_1^*, so by the lemma, $u^* I_i \subseteq (g_1^*)$. Thus we have

$$u I \subseteq (S^+ R) \cap ((g_1) + P_i R)$$

for all i. Since every prime of R contains either $S^+ R$ or some $P_i R$, it follows that

$$u I \subseteq \sqrt{(g_1)}$$

as desired. ‖

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