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REMARKS ON IDEALS AND RESOLUTIONS (*) (**)

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0. Introduction.

In our work on the lifting problem we have been led to a detailed study of finite free resolutions. We needed (and still need) to know both about the generic properties of resolutions—that is, the equational properties that all resolutions share, and the properties of the resolutions of generic ideals and modules—and also about the special properties of resolutions of certain particular types of ideals. In this paper, we present a result of each of these two types.

In section 1, we prove a result about the minimal generation of ideals I in a regular local ring R such that R/I is Gorenstein. We are grateful to R. Zibman for his able programming of the Brandeis University PDP-10 computer to produce the examples on which the second half of section 1 is based.

In section 2 we turn to the generic situation, and show how to construct a minimal free resolution of a generic module. At the same time we show how to redo the theory of the Eagon-Northcott complex resolving a certain ideal of minors connected with a matrix) in a basis-free manner.

The complexes associated with the lower order minors of a matrix still need further study. We feel that better understanding of the complexes attached to the maximal minors of a matrix may be of great help. The construction of the complexes associated with the maximal minors of a matrix that we have indicated here has suggested an approach to the symmetrization of the complexes given in [3, 4].

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1. Gorenstein ideals.

The lifting problem of Grothendieck is the following: Given a regular local ring S with a regular factor ring $R = S/(x)$, and an R-module $\bar{M}$, does there exist an S module M such that x is a nonzero-divisor on M and $M/xM \approx \bar{M}$?

In the course of our work on this problem, it became clear that lifting R-modules of the form R/I, where R/I is a Gorenstein ring, might be particularly difficult. (Recall that a local ring R is Gorenstein [1] if and only if it is Macaulay and some system of parameters generates an irreducible ideal.) If J is an ideal of a local ring R, we will say that J is a Gorenstein ideal if R/J is a Gorenstein ring. If I is a Gorenstein ideal and $\mathrm{hd}_R(R/I) = n$, then it easily follows that I cannot be minimally generated by fewer than n elements. If the minimal number of generators of I is exactly n, then I can be generated by an R-sequence, and lifting R/I is trivial. (See [6] for a discussion of this and related results.) If $n = 2$, this is the only possibility; an easy observation due to Serre [1] shows that if I is an ideal in a regular local ring R such that R/I has homological dimension 2, then I is Gorenstein if and only if I is generated by an R sequence, and in particular can be generated by two elements. Thus, for the lifting problem, the case in which I is Gorenstein, has homological dimension 3, and I cannot be generated by three elements, is the first case that presents difficulties. Our study of the lifting problem showed that if, in this case, I is minimally generated by five or more elements, then the problem of lifting the R-module R/I could be reduced to a different lifting problem, which seemed more tractable. Thus we were pleased to discover that there *were* no ideals I satisfying the conditions above and, in addition, requiring exactly four generators. More generally:

THEOREM 1.1: Let R be a regular local ring, and suppose that $I \subset R$ is a Gorenstein ideal with $\mathrm{hd}_R R/I = n$. Then the minimal number of generators of I can *not* be $n + 1$.

What *can* the minimal number of generators of a Gorenstein ideal be? Aside from Theorem 1.1, and the result of Serre on ideals I such that $\mathrm{hd}_R R/I = 2$, we know of no restrictions. However, there is some experimental evidence for the existence of some further limitations, which we will now describe.

The following lemma shows that one can restrict one's attention to ideals primary to the maximal ideal.

LEMMA 1.2: *a*) Let R be a local ring with maximal ideal $\mathfrak{m}$, and let I be an ideal of R. Suppose that the minimal number of generators of I

is g. Let $x \in \mathfrak{m}$ be a nonzerodivisor for R/I. Then the minimal number of generators of (I, x) is $g+1$.

b) With R as above, suppose that $\dim R = d$, and that for every ideal I of R, $\dim I + \operatorname{ht} I = \dim R$. If I is a Gorenstein ideal of R such that $\operatorname{ht} I = n$ and I requires $n+k$ generators, then there is a Gorenstein ideal J of R, with $\operatorname{ht} J = d$, minimally generated by $d+k$ elements.

PROOF: *a*) Let $x_1, \dots, x_g$ be a minimal set of generators of I; we will show that $x_1, \dots, x_g, x$ is a minimal set of generators of (I, x). To show this, we must show that if $\sum r_i x_i + rx = 0$, with $r_i, r \in R$, then the elements r_i, r are all in the maximal ideal $\mathfrak{m}$. Suppose, then, that $\sum r_i x_i + rx = 0$. Since x is not a zero divisor for R/I, we have $r \in I \subset \mathfrak{m}$ and, since $x \in \mathfrak{m}$, $rx \in \mathfrak{m}I$. Thus $\sum r_i x_i = -rx \in \mathfrak{m}I$ and, since $x_1, \dots, x_g$ is a minimal set of generators of I, it follows that the elements r_i are all in $\mathfrak{m}$.

b) If $s_1, \dots, s_{d-n}$ is a maximal R/I-sequence, then $(I, s_1, \dots, s_{d-n})$ is a Gorenstein ideal which is primary to the maximal ideal and requires $(d-n)+n+k = d+k$ generators, by part *a*). ∎

It is fairly easy to survey the Gorenstein ideals I which are primary to the maximal ideal $\mathfrak{m}$ of a Gorenstein local ring R. For this we will use various results about Gorenstein rings whose proofs may be found in [1].

If $s_1, \dots, s_d$ is any maximal R sequence then I must contain the ideal $(s_1^k, \dots, s_d^k)$ for some k, since we are assuming that I is $\mathfrak{m}$-primary. On the other hand, $\bar{R} = R/(s_1^k, \dots, s_d^k)$ is a zero-dimensional Gorenstein ring, and thus self-injective; in fact $\bar{R}$ is the $\bar{R}$-injective envelope of the socle of $\bar{R}$, which is simple. For any $I \supseteq (s_1^k, \dots, s_d^k)$, $\operatorname{Hom}_{\bar{R}}(R/I, \bar{R})$ is an injective R/I-module with simple socle. Thus if R/I is Gorenstein, $\operatorname{Hom}_{\bar{R}}(R/I, \bar{R}) \cong R/I$. Since $R/I \cong \operatorname{Hom}_{\bar{R}}(R/I, \bar{R}) \cong \operatorname{Hom}_{\bar{R}}(R/I, \bar{R}) \subset \bar{R}$, there is an element $x \in R$ such that the ideal $(\bar{x})$ in $\bar{R}$ has, as its annihilator in R, precisely the ideal I. This means, then, that $I = (s_1^k, \dots, s_d^k) : x$.

Conversely suppose $x \in R$ is any element, and let $\bar{x}$ be its image in $\bar{R}$. Set $I = \operatorname{ann}_R(\bar{x}) = (s_1^k, \dots, s_d^k) : x$. Then $I \subset (s_1^k, \dots, s_d^k)$ and $\bar{I} = I/(s_1^k, \dots, s_d^k)$ is an ideal of $\bar{R}$. For any ideal J in a zero-dimensional Gorenstein ring $\bar{R}$, $\operatorname{ann}_{\bar{R}}(\operatorname{ann}_{\bar{R}} J) = J$ so that, with $J = \bar{I}$, we have

$$\operatorname{Hom}_R(R/I, \bar{R}) = \operatorname{Hom}_{\bar{R}}(\bar{R}/\bar{I}, \bar{R}) = \operatorname{ann}_{\bar{R}} \bar{I} = \operatorname{ann}_{\bar{R}}(\operatorname{ann}_{\bar{R}}(\bar{x})) = (\bar{x}) .$$

Since $(\bar{x}) \approx R/I$, we have

$$\operatorname{Hom}_{\bar{R}}(R/I, \bar{R}) \approx R/I .$$

Thus R/I is self-injective, so I is a Gorenstein ideal. We have proved:

PROPOSITION 1.3: Let R be a local Gorenstein ring with maximal ideal $\mathfrak{m}$, and let $s_1, \ldots, s_d \in \mathfrak{m}$ be a maximal R-sequence. Then every $\mathfrak{m}$-primary Gorenstein ideal of R has the form $(s_1^k, \ldots, s_d^k): x$ for some integer k and element $x \in R$. ∎

Using this proposition, it is relatively easy to get examples of Gorenstein ideals. In fact, if we restrict our attention to homogeneous ideals in a ring of the form $R = K[x_1, \ldots, x_d]$ with K a field, then producing Gorenstein ideals which are primary to $(x_1, \ldots, x_d)$ and finding their minimal number of generators is a matter of linear algebra. Because the systems of linear equations involved become too large for calculation by hand, we have used Brandeis University's PDP-10 computer to give us a range of examples of homogeneous Gorenstein ideals of height three in the ring $R = K[x_1, x_2, x_3]$, where K is the field of rational numbers. In this case, three-generator Gorenstein ideals are easy to find—they are just ideals generated by R sequences.

Gorenstein ideals requiring exactly four generators are ruled out by Theorem 1.1 (or the homogeneous version of 1.1). Because of restrictions in the size of the PDP-10 core storage, we thus far looked primarily for ideals of the form $I = (x_1^4, x_2^4, x_3^4): x$. Though we have examined many examples, we have found only ideals whose minimal number of generators is 5, 7 and 9. It seems plausible that we would quickly find ideals requiring more than nine generators by working modulo higher powers of x_1, x_2, x_3. But the reason for the non-appearance of ideals requiring exactly six or eight generators remains a mystery.

Here are some of the examples the computer has obtained:

5 generators: $(x_1^4, x_2^4, x_3^4): (x_1x_2 + x_1x_3 + x_2x_3)$

$$= (x_1^4, x_2^4, x_3^4, x_1x_2^3 + x_2^3x_3 - x_1^3x_2 - x_1^3x_3 + x_1^2x_2x_3 - x_1x_2^2x_3$$
$$x_1x_3^3 + x_2x_3^3 - x_1^3x_2 - x_1^3x_3 + x_1^2x_2x_3 - x_1x_2x_3^2)$$

7 generators: $(x_1^4, x_2^4, x_3^4): (x_1^3 + x_1x_2x_3)$

$$= (x_1^3, x_2^4, x_3^4, x_1^2x_3^2 - x_2x_3^3, x_1x_3^3, x_1x_2^3, x_1^2x_2^2 - x_2^3x_3)$$

9 generators: $(x_1^4, x_2^4, x_3^4): (x_1^3 + x_2^3 + x_3^3 + x_1x_2x_3)$

$$= (x_1^4, x_2^4, x_3^4, x_1^3x_2 - x_2x_3^3, x_1x_2^3 - x_1x_3^3, x_2^3x_3 - x_2^3x_3,$$
$$x_1x_2^3 + x_1^2x_2x_3 - x_2^2x_3^2, x_1^2x_2^2 - x_1^3x_3 - x_1x_2x_3^2,$$
$$x_1^3x_2 + x_1x_2^2x_3 - x_1^2x_3^2).$$

It is perhaps worth remarking that though we have no example of a height 3 Gorenstein ideal requiring $3 + 5 = 8$ generators, it is easy to see that

$$\begin{aligned}(x_1^2, x_2^2, x_3^2, x_4^2)&: (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)\\ &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 - x_3x_4,\\ &\quad x_1x_2 - x_2x_4, x_1x_2 - x_1x_4,\\ &\quad x_1x_2 - x_2x_3, x_1x_2 - x_1x_3)\end{aligned}$$

is a Gorenstein ideal of height 4 requiring $4 + 5 = 9$ generators.

We now turn to the proof of Theorem 1.1. We will use Serre's characterization [1] of Gorenstein factor rings of regular local rings:

THEOREM 1.4 (SERRE): Let R be a regular local ring, and I an ideal of R of height n. Then R/I is Gorenstein if and only if $\operatorname{hd}_R R/I = n$ and $\operatorname{Ext}_R^n(R/I, R) = R/I$. ∎

We will show that if I is a Gorenstein ideal of height n which is generated by $n+1$ elements, then I can be generated by an R-sequence, which must consist, then, of n elements. For this we will use the following lovely characterization of ideals generated by R-sequences, which is due to Gulliksen [13].

THEOREM 1.5 (GULLIKSEN): Let R be a local ring, and I an ideal of R of finite homological dimension. Suppose $I = (g_1, \ldots, g_k)$, and let $\mathbb{C}$ be the Koszul complex associated to $g_1, \ldots, g_k$. Then I can be generated by an R-sequence if and only if $H_1(\mathbb{C})$ is a free R/I-module. ∎

PROOF OF THEOREM 1: Suppose $I = (g_1, \ldots, g_{n+1})$, and let $\mathbb{C}$ be the Koszul complex associated to $g_1, \ldots, g_{n+1}$. Since depth $I = \operatorname{ht} I = n$, the only nonvanishing homology group of $\mathbb{C}$ is $H_1(\mathbb{C})$. Now it is well-known (see, for example, [5]) that $H_1(\mathbb{C})$ is isomorphic to the first nonvanishing $\operatorname{Ext}_R^k(R/I, R)$. Since depth $I = n$, this is $\operatorname{Ext}_R^n(R/I, R)$, and by Theorem 1.4 this is R/I. Thus $H_1(\mathbb{C}) \simeq R/I$, so by Theorem 1.5, I can be generated by an R-sequence. ∎

2. Generic resolutions.

In this section we will describe a complex which is an approximate minimal free resolution of an arbitrary torsion module in the same sense that the classical Koszul complex is an approximate minimal resolution of an arbitrary cyclic module. More precisely, the complex that we will describe depends (in a certain sense functorially) on a given presentation of the torsion module. The complex yields a minimal free resolution of any «generic» torsion module, and is therefore grade sensitive.

Our «generic resolution» is very closely related to the complex defined by Eagon and Northcott in [11]. In fact, the complex we describe may be thought of as arising from the complexes of Buchsbaum [3] and Buchsbaum-Rim [8] by the process of «symmetrization» suggested by the complex of Eagon and Northcott and made explicit by Gover in [12]. Thus at the same time as we describe the generic resolution, we will show how to obtain an intrinsic description of the Eagon-Northcott complex, and give a proof of its properties which does not require the rather complicated induction used in [11].

Our treatment of the Eagon-Northcott complex owes a considerable debt to the one given by E. Gover in his thesis, [12] which includes a clean description of the differential of the complex.

Now to define the complexes about which we have been speaking. Suppose that R is a commutative ring, A and B are R-modules, and $f \in A^* \otimes_R B$. Then we may regard f as a bihomogeneous element of bidegree 1,1 in the bigraded algebra

$$E = \wedge(A^*) \otimes S(B)\,,$$

where $\wedge(A^*)$ is the exterior algebra of A^* and $S(B)$ is the symmetric algebra of B.

Since $S(B)$ is a commutative algebra and $\wedge(A^*)$ is anticommutative, we have $f^2 = 0$ so we may regard E as a bigraded differential R-algebra whose differential is multiplication by f.

As a complex of R-modules, E is just a direct sum of complexes of the form

$$\ldots \to \overset{k}{\wedge} A^* \otimes S_j(B) \xrightarrow{\partial_f} \overset{k+1}{\wedge} A^* \otimes S_{j+1}(B) \to \ldots\,.$$

If $f = \sum \alpha_i \otimes b_i$ with $\alpha_i \in A^*$ and $b_i \in B$, then

$$\partial_f(\alpha_1' \wedge \ldots \wedge \alpha_k' \otimes b_1' \ldots b_j') = \sum_i \alpha_i \wedge \alpha_1' \wedge \ldots \wedge \alpha_k' \otimes b_i b_1' \ldots b_j'\,.$$

The complexes we are seeking are simply the duals of certain of these complexes, suitably augmented. We will restrict ourselves to the case in which A and B are finitely generated free R-modules. In this situation, there is a natural isomorphism

$$\mathrm{Hom}_R(A, B) \to A^* \otimes_R B$$

so that if $f\colon A \to B$ is a morphism, we may view f as an element of $A^* \otimes B$.

Suppose now that A has rank m and B has rank n, and $f \in \operatorname{Hom}(A, B)$. We define the generic resolution $G(f)$ associated to f to be the complex

$$G(f)\colon 0 \to S_{m-n-1}(B^*) \otimes \overset{m}{\wedge} A \xrightarrow{d_f} S_{m-n-2}(B^*) \otimes \overset{m-1}{\wedge} A \to \dots$$
$$\dots \xrightarrow{d_f} S_0(B^*) \otimes \overset{n+1}{\wedge} A \xrightarrow{\varepsilon} A \xrightarrow{f} B$$

where d_f is the composite map:

$$S_j(B^*) \otimes \overset{k}{\wedge} A \overset{\approx}{\to} [S_j(B) \otimes \overset{k}{\wedge} A^*]^* \xrightarrow{\partial_f^*} [S_{j-1}(B) \otimes \overset{k-1}{\wedge} A^*]^* \overset{\approx}{\to} S_{j-1}(B^*) \otimes \overset{k-1}{\wedge} A\,,$$

and ε is the composite:

$$S_0(B^*) \otimes \overset{n+1}{\wedge} A \overset{\approx}{\to} \overset{n}{\wedge} B^* \otimes \overset{n+1}{\wedge} A \xrightarrow{\overset{n}{\wedge} f \otimes 1} \overset{n}{\wedge} A^* \otimes \overset{n+1}{\wedge} A \to A$$

where the left hand isomorphism comes from an arbitrary choice of an isomorphism $S_0(B^*) = R \approx \overset{n}{\wedge} B^*$, and the right hand map is the restriction of the map

$$\wedge A^* \otimes \wedge A \to \wedge A$$

coming from the canonical map $A^* \otimes A \to R$. (For this and other details of multilinear algebra, Bourbaki [2] is an excellent source. A complete exposition of this may also be found in [4].)

The equations $\varepsilon d_f = 0 = f\varepsilon$ follow easily from the fact that $\overset{n+1}{\wedge} f = 0$.

Using another summand of the algebra E we may recover the Eagon-Northcott complex [11] associated to a map $f\colon A \to B$ where, as before, A and B are free R modules of rank m and n respectively. This is

$$E-N(f)\colon 0 \to S_{m-n}(B^*) \otimes \overset{m}{\wedge} A \xrightarrow{d_f} S_{m-n-1}(B^*) \otimes \overset{m-1}{\wedge} A \xrightarrow{d_f} \dots \xrightarrow{d_f}$$
$$\to S_0(B^*) \otimes \overset{n}{\wedge} A \xrightarrow{\overset{n}{\wedge} f} \overset{n}{\wedge} B\,.$$

Here the maps labeled d_f are as in the generic resolution, above, while to define the last map we simply use the fact that $S_0(B^*) = R$. Again the equation $\overset{n}{\wedge} f d_f = 0$ follows from the fact that $\overset{n+1}{\wedge} f = 0$.

We note that the complexes $G(f)$ and $E-N(f)$ both have length $m-n+1$.

The exactness properties of these two complexes depend on the depth of the ideal $I_n(f)$ (the n-th Fitting invariant of the cokernel of f) which is defined to be:

$$I_n(f) = \operatorname{im}\left(\overset{n}{\wedge} A^* \otimes \overset{n}{\wedge} B \xrightarrow{\varphi} R\right)$$

where φ is the map associated to

$$\overset{n}{\wedge} f \colon \overset{n}{\wedge} A \to \overset{n}{\wedge} B \,.$$

We remark that (under the assumption that rank $B = n$) $I_n(f)$ has the same radical as the annihilator of Coker f.

We are now ready to state the main theorem:

THEOREM 2.1: Let R be a commutative noetherian ring, and suppose that $f\colon A \to B$ is a map of free R-modules A and B whose ranks are m and n respectively. Let $\mathbf{C}$ be any of the complexes

$$G(f),\, G(f)^*,\, E - N(f),\, E - N(f)^*$$

where $*$ denotes R-linear dual. Then if $\operatorname{depth} I_n(f) = d \leqslant m - n + 1$, the last non-vanishing homology group of $\mathbf{C}$ is $H_{m-n+1-d}$.

REMARK: It is true that $d \leqslant m - n + 1$ for *any* map $f\colon A \to B$; this follows, for example, from [8] and [9] (the first result of this kind goes back to Macaulay [14]). Here is a sketch of an easy alternative proof. We will show in the course of the proof of Theorem 2.1 that $G(f)$ is a free resolution of Coker f whenever $d \geqslant m - n + 1$ so that, in this case, hd (Coker $f) \leqslant m - n + 1$. But

$$\operatorname{depth} I_n(f) = \operatorname{depth} \operatorname{ann}(\operatorname{Coker} f)$$

since these ideals have the same radical. However, from

$$\operatorname{depth} \operatorname{ann}(\operatorname{Coker} f) > m - n + 1$$

would follow

$$\operatorname{hd}(\operatorname{Coker} f) > m - n + 1\,,$$

a contradiction.

Before proving Theorem 2.1, we make note of a consequence: It is known [10] that if S is any ring, $\{X_{ij}\}$ are mn indeterminates with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and R is the polynomial ring $R = S[\{X_{ij}\}]$, then the « generic homomorphism » $f\colon R^m \to R^n$ whose matrix is (X_{ij}) satisfies

$$\operatorname{depth} I_n(f) = m - n + 1 .$$

Thus the complex $G(f)$ and $E - N(f)$ associated to f are exact.

To prove Theorem 2.1, we will use a result from [7]. For any map φ of free modules, the rank of φ is defined to be the largest integer r such that $\overset{r}{\wedge}\varphi \neq 0$. If $r = \operatorname{rank}\varphi$, we set $I(\varphi) = I_r(\varphi)$. The result from [7] is:

THEOREM 2.2: Let R be a noetherian ring with a connected spectrum, and let

$$\mathbf{C}\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \dots \to F_1 \xrightarrow{\varphi_1} F_0$$

be a complex of free R-modules.

Then $\mathbf{C}$ is exact if and only if for each k

1) $\operatorname{rank} \varphi_k + \operatorname{rank} \varphi_{k-1} = \operatorname{rank} F_{k-1}$

2) $\operatorname{depth} I(\varphi_k) \geqslant k$.

COROLLARY: Let R be any noetherian ring, and let

$$\mathbf{C}\colon 0 \to F_n \xrightarrow{\varphi_n} \dots \to F_1 \xrightarrow{\varphi_1} F_0$$

be a complex of free modules. Suppose that for every prime P with $\operatorname{depth} P_P < n$, $\mathbf{C}_P$ is exact. Then $\mathbf{C}$ is exact.

PROOF: This is an immediate application of Theorem 2, once one notes that if I is an ideal of R with $\operatorname{depth} I < k$, then there is a prime ideal $P \supset I$ such that $\operatorname{depth} P_P < k$. ∎

REMARK: The Corollary also follows immediately from the « Lemme d'Acyclicité » of Peskine and Szpiro [15].

PROOF OF THEOREM 2.1: We will show that if $\operatorname{depth} I_n(f) \geqslant d$, then the complex

$$0 \to \mathbf{C}_{m-n+1} \to \dots \to \mathbf{C}_{m-n+1-d}$$

is exact, so that $H_{m-n+1}(\mathbf{C}) = \dots = H_{m-n+2-d}(\mathbf{C}) = 0$. From this it follows by standard arguments that if $\operatorname{depth} I_n(f) = d$, then $H_{m-n+1-d}(\mathbf{C}) \cong \operatorname{Ext}^d(M, R) \neq 0$ (see [7, 8, 10]), so the theorem will be complete.

It now clearly suffices to prove that $\mathbb{C}_P$ is exact for all primes P such depth $P_P < d$. Since depth $I_n(f) \geqslant d$, we must have $I_n(f_P) = R$ for all these primes, so that f_P is a split epimorphism and $A_P = A'_P \oplus B_P$ for some free R-module A'. Since the definition of $\mathbb{C}$ is clearly invariant under localization, we are reduced to proving the following:

PROPOSITION 2.3: Let A' and B be free R-modules of ranks $m-n$ and n respectively and let

$$f\colon A' \oplus B \to B$$

be the projection onto the second factor. Then if C is one of the complexes of Theorem 2.1 associated to f, C is split exact.

PROOF: Split exactness at the ends of the maps ε, ε^*, $\overset{n}{\wedge} f$, $\overset{n}{\wedge} f^*$, f, f^* is easy to see using the usual decomposition

$$\overset{k}{\wedge}(A' \oplus B) = \sum_{i+j=k} \overset{i}{\wedge} A' \otimes \overset{j}{\wedge} B .$$

Thus it suffices to prove the split exactness of the complexes

$$(*) \qquad \ldots \to \overset{k-1}{\wedge} A^* \otimes S_{j-1}(B) \to \overset{k}{\wedge} A^* \otimes S_j(B) \to \overset{k+1}{\wedge} A^* \otimes S_{j+1}(B) \to \ldots$$

at terms where $k \geqslant 1$. These complexes, of course, are summands of the graded differential algebra $E = \wedge A^* \otimes SB$ associated to the map f.

We will show that $H_{kj}(E) = 0$ for all $k \geqslant 1$, so that the complexes $(*)$ are exact at terms whose $k \geqslant 1$. Splitness then follows from the fact that the last term at the right hand end of each of the complexes in $(*)$ is free.

In order to compute the homology of E, it is convenient to change our point of view. Since $\wedge A^* \otimes_R S(B)$ is the exterior algebra over $S(B)$ of $A^* \otimes_R S(B)$, we may regard E as an exterior algebra over the graded ring $S(B)$. Moreover, the differential, ∂_f, when viewed in this light, is nothing but the differential of the ordinary (graded) Koszul complex, defined over $S(B)$, of the map $A \otimes S(B) \to S(B)$ induced by $f\colon A \to B$.

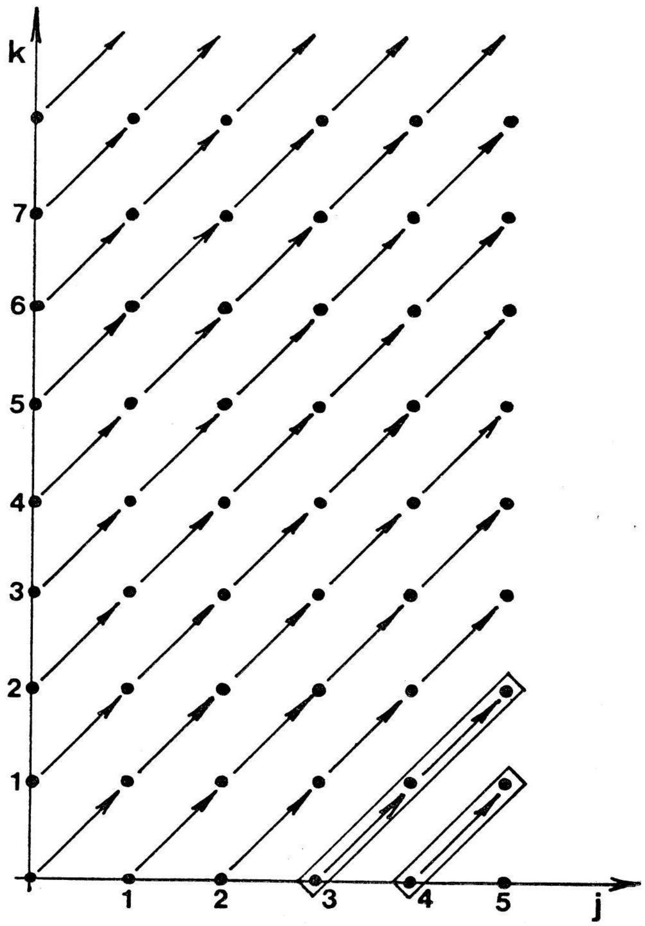
To make this explicit, let $\{b_1, \ldots, b_n\}$ be a basis for B, so that $S(B) = R[b_1, \ldots, b_n]$. $A \otimes S(B)$ is a free $S(B)$-module on m generators, and the projection $A \to B$ induces the map of free $S(B)$-modules with matrix

$$A \otimes S(B) \xrightarrow{(\overbrace{0, \ldots, 0}^{m-n}, b_1, \ldots, b_n)} S(B) .$$

An easy computation of the (bi-graded) homology module of such a Koszul complex shows that the only nonzero homology modules are

$$H_{jk} \text{ with } k \leqslant m \text{ and } j = 0 .$$

The picture of E below may help to explain these calculations:



$\bigwedge^{j} A^* \otimes S_k A$ for rank $A = 5 = m$

rank $B = 3$

(The dots are the nonzero terms of E.) The nonzero homology all occurs in the shaded region, while the complexes from which G, and $E - N$ are built are enclosed in rectangles.

REMARK: We have recently succeeded in constructing several complexes of length $m-n+1$ using the algebra and its dual. In this way, one can obtain, starting with a map $f: A \to B$ of free modules of ranks m and n respectively generic resolutions of the modules:

i) $\overset{m-n+1}{\wedge}$ (Coker f^*)

ii) S_k (Coker f) for $k \leq m-n+1$.

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REFERENCES

[1] H. BASS, *On the ubiquity of Gorenstein rings*, Math. Zeitschr., 82 (1963), 8-28.

[2] N. BOURBAKI, *Elements de Mathématiques, Algèbre*, Chapter III, Hermann, 1970.

[3] D. BUCHSBAUM, *A generalized Koszul complex, I*, Trans. Amer. Math. Soc., 111 (1964), 183-197.

[4] D. BUCHSBAUM, *Complexes associated with the minors of a matrix*, Symposia Math., IV (1970, 255-283.

[5] D. BUCHSBAUM, *Lectures on regular local rings*, Springer Lecture notes in Mathematics, no. 86 (1968).

[6] D. BUCHSBAUM and D. EISENBUD, *Lifting modules and a theorem on finite free resolutions*, to appear in Proc. Utah Conf. on Ring Theory, Springer lecture notes in Mathematics.

[7] D. BUCHSBAUM and D. EISENBUD, *What makes a complex exact?*, to appear in J. of Alg.

[8] D. BUCHSBAUM and D. RIM, *A generalized Koszul complex. - II: Depth and multiplicity*, Trans. Amer. Math. Soc., 111 (1964), 197-225.

[9] J. EAGON, *Thesis*, University of Chicago.

[10] J. EAGON and M. HOCHSTER, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math., 93 (1971), 1020-1059.

[11] J. EAGON and D. NORTHCOTT, *Ideals defined by matrices and a certain complex associated to them*, Proc. Royal Soc., A 269 (1962), 188-204.

[12] E. GOVER, *Generalized local complete intersections*, Thesis, Brandeis University, 1970.

[13] T. GULLIKSEN, *Homological invariants of Local Rings*, Queen's Math. Preprint no. 12.

[14] F. S. MACAULAY, *Algebraic theory of modular systems*, Cambridge Mathematical Tract no. 42, Cambridge, 1953.

[15] L. PESKINE and C. SZPIRO, *Thesis*, to appear in Pub. I.H.E.S., Paris.