JOURNAL OF ALGEBRA 25, 259–268 (1973)

What Makes a Complex Exact?*

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Received August 1, 1971

INTRODUCTION

In this paper we give a simple criterion for the exactness of a finite complex

$$\mathbb{A}: 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated projective modules over a commutative noetherian ring, and for the exactness of such a complex when tensored with a finitely generated module. The criterion consists of a condition on the ranks of the φ_i which is familiar from the theory of finite dimensional vectorspaces, and a second, arithmetic, condition which involves only one of the maps φ_i at a time.

In vectorspace theory, the nonvanishing minors of maximal size of a matrix play a crucial role in determining the solutions of the linear equations that correspond to the matrix. The theory of linear equations over a commutative ring is complicated by the fact that the ideal generated by the nonvanishing minors of maximal size need not contain a unit. It is the depths of the ideals of nonvanishing minors of maximal size of the φ_k which enters the second condition.

The criterion, which is precisely stated at the beginning of §1, can be used to simplify the theory of generalized Koszul complexes of Buchsbaum–Rim and Eagon–Northcott. It has also proved useful in an attack on the lifting problem of Grothendieck [2].

PRELIMINARIES

Throughout this paper, rings are assumed commutative with identity.

Suppose that R is a ring, $\varphi: F \to G$ is a map of projective R-modules, and M is any R-module. We define the rank of φ, written rank φ, to be the

* This paper was written while the authors were partially supported by NSF GP 23119.

largest integer k such that $\wedge^k \varphi \neq 0$, $\wedge^{k+1} \varphi = 0$. When F and G are free, this is the same thing as the size of the largest nonvanishing minor of a matrix corresponding to φ. Similarly, we define the rank of φ on M, written rank(φ, M), to be the largest k with $\wedge^k \varphi \otimes M \neq 0$. When F and G are free, this is the same as the size of the largest minor r of a matrix corresponding to φ such that r is not contained in the annihilator of M. (As usual, if ψ is a map and M is a module, we write $\psi \otimes M$ for $\psi \otimes 1_M$.)

Since any map f: $A \to B$ yields, in the obvious way, a map $B^* \otimes A \to R$, the map $\wedge^k \varphi$: $\wedge^k F \to \wedge^k G$ yields a map $(\wedge^k G)^* \otimes \wedge^k F \to R$ induced by $\wedge^k \varphi$. We define $I(\varphi)$ to be the image of this map for $k = \operatorname{rank} \varphi$. Similarly, if M is an R-module, we define $I(\varphi, M)$ to be the image of the above map where $k = \operatorname{rank}(\varphi, M)$. When F and G are free, it is easy to see that $I(\varphi)$ is the ideal generated by the $k \times k$ minors of a matrix corresponding to the map φ. The rank of a free module is defined to be the cardinality of a basis for the module; over a commutative ring, this is well defined.

Now suppose R is noetherian. If I is any ideal of R, and if $I \neq R$, we let depth(I, M) be the length of a maximal M-sequence contained in I. If $I = R$, we set depth$(I, M) = \infty$. We note that depth(I, M) is also the smallest integer k such that $\operatorname{Ext}^k(R/I, M) \neq 0$. (See, for example, [6, p. 100].

The main fact we will use about depth is that if I is an ideal of R and $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then there are relations between the numbers depth(I, A), depth(I, B), and depth(I, C), which may be seen by examining the long exact sequence in $\operatorname{Ext}(R/I, -)$ to which the short exact sequence $0 \to A \to B \to C \to 0$ gives rise. For example, if depth$(I, A) <$ depth(I, B), then depth$(I, C) =$ depth$(I, A) - 1$.

1. The Criterion

THEOREM. *Let R be a commutative noetherian ring, $M \neq 0$ a finitely generated R-module, and*

$$\mathbb{A}: 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

a complex of free R-modules such that for all $k = 1,..., n$, $\varphi_k \otimes M \neq 0$. Then $\mathbb{A} \otimes_R M$ is exact if and only if for $k = 1,..., n$,

(a) $\operatorname{rank}(\varphi_{k+1}, M) + \operatorname{rank}(\varphi_k, M) = \operatorname{rank} F_k$

(b) *$I(\varphi_k, M)$ contains an M-sequence of length k or $I(\varphi_k, M) = R$.*

A useful special case is:

COROLLARY 1. *Let R and $\mathbb{A}$ be as in the theorem. Then $\mathbb{A}$ is exact if and only if*

(a) $\operatorname{rank} \varphi_{k+1} + \operatorname{rank} \varphi_k = \operatorname{rank} F_k$,

(b) $I(\varphi_n)$ *contains an R-sequence of length k or* $I(\varphi_n) = R$. ▌

It is not difficult to extend the theorem to deal with complexes of projective modules, as long as the ranks of the projectives are well defined. The rank of a projective module G is defined to be the rank of the free R_P-module G_P for any maximal ideal P, providing that these all coincide. In particular, if the noetherian ring R has no idempotents other than 0 and 1, these ranks always do coincide, so that in this case the rank of every projective is well defined [8, Theorems 7.8 and 7.12].

COROLLARY 2. *Let R be a commutative noetherian ring, let* $M \neq 0$ *be an R-module, and let*

$$\mathbb{A}: 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

be a complex of finitely generated projective R-modules of well defined rank. Then $\mathbb{A} \otimes M$ *is exact if and only if for all* $k = 1,..., n$

(a) $\operatorname{rank}(\varphi_{k+1}, M) + \operatorname{rank}(\varphi_k, M) = \operatorname{rank} F_k$ *and*

(b) $I(\varphi_k, M)$ *contains an M-sequence of length k or* $I(\varphi_k, M) = R$.

Proof of Corollary 2. If I is an ideal that contains no M-sequence of length k, then there is a prime $P \geqslant I + \operatorname{ann}_R M$ such that P_P contains no M_P sequence of length k [6, Theorem 125]. Thus we may localize and assume that the F_k are free. An application of the theorem completes the proof. ▌

Remarks. Suppose that R and $\mathbb{A}$ are as in the theorem, and that $\mathbb{A}$ is exact. Recall that for any ideal I of R, $\sqrt{I}$ denotes the ideal $\{r \in R \mid r^m \in I$ for some $m\}$. Lemma 1, below, combined with an easy localization argument, shows that for each k,

$$\sqrt{I(\varphi_{k+1})} \geqslant \sqrt{I(\varphi_k)}.$$

Actually, much more is true; there exist ideals H_k for $k = 0,..., n$, such that

$$I(\varphi_k) = H_k \cdot H_{k-1}, \qquad k = 1,..., n$$

and

$$\sqrt{H_k} = \sqrt{I(\varphi_{k+1})} \qquad k = 2,..., n-1.$$

The ideals H_k are generated by elements which are intimately involved in the structure of the resolution $\mathbb{A}$; some of this is described in [2] (in the notation of [2], H_k is the ideal generated by the elements $\alpha_k(J)$). Reference [2a] will contain more in this direction.

2. Proof that Conditions (a) and (b) imply Exactness

We now prove the easier half of the theorem, namely, the sufficiency of conditions (a) and (b). The technique is to reduce the length of $\mathbb{A}$ till it is less than the depth of M (Lemmas 1 and 2) and then use an idea of Peskine and Szpiro (Lemma 3).

Lemma 1 seems to belong to the folklore.

LEMMA 1. *Let R be a commutative ring, and let $\varphi: F \to G$ be a map of finitely generated free R-modules. Then* coker φ *is projective and has well-defined rank if and only if $I(\varphi) = R$.*

Proof. $\Leftarrow$: A finitely presented module is projective if and only if it is locally free, so we may suppose that R is local. Thus if rank $\varphi = k$, then $I(\varphi) = R$ if and only if some $k \times k$ minor of φ is a unit. Thus we may choose bases of F and G so that the matrix of φ takes the form

$$\varphi = \left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right),$$

where A is a $k \times k$ matrix with unit determinant. Since A represents an isomorphism, we may change bases so that A is the $k \times k$ identity matrix

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

By elementary row and column transformations, we may replace B and C by 0 matrices. If now D were not 0, the rank of φ would be bigger than k; thus $D = 0$ also. The matrix of φ now has the form

$$\varphi = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right);$$

it is clear that coker φ is free of rank (rank G — rank φ).

$\Rightarrow$: Since coker φ has well-defined rank, we may assume that R is local. If coker φ is projective, it is free, and we have $F \simeq F' \oplus F''$, $G \simeq F'' \oplus \operatorname{coker} \varphi$ $\varphi = 1_{F''} \oplus (F' \to^0 \operatorname{coker} \varphi)$; it is clear that $I(\varphi) = R$. ▮

The next lemma handles the nonarithmetic part of the theorem.

LEMMA 2. *Let R be a commutative ring, $M \neq 0$ an R module, and*

$$\mathbb{B}: F \xrightarrow{\varphi} G \xrightarrow{\psi} H$$

a complex of finitely generated free R-modules such that $I(\varphi, M) = I(\psi, M) = R$. Then $\mathbb{B} \otimes M$ is exact if and only if $\operatorname{rank}(\varphi, M) + \operatorname{rank}(\psi, M) = \operatorname{rank} G$.

Proof. We may begin by factoring out the annihilator of M. Thus we can assume that M is faithful and replace $I(-, M)$ by $I(-)$ and $\operatorname{rank}(-, M)$ by $\operatorname{rank}(-)$ in the statement of the lemma. Also, we may assume that R is local. As in Lemma 1 we may choose bases for F and G which diagonalize φ. Choosing new bases for a complement in G of the image of φ, and for H, we may write $\mathbb{B}$ in the form

$$\mathbb{B}: F \xrightarrow{\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)} G \xrightarrow{\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right)} H.$$

It is now evident that $\mathbb{B} \otimes M$ is exact if and only if the size of the identity matrix in the upper left corner of φ and the size of that in the lower right corner of ψ add up to the rank of G. ▮

Proof of the sufficiency of the conditions of the theorem. As usual, we begin by localizing, and we let J be the maximal ideal of R. Suppose $\operatorname{depth}(J, M) = l$. Then for $k > l$, $I(\varphi_k, M) = R$, this being the only way hypothesis (b) can be satisfied. We set $F_l' = \operatorname{coker} \varphi_{l+1}$, which is free by Lemma 1, and let $\varphi_l' : F_{l'} \to F_l$ be the map induced by φ_l. The complex $\mathbb{A}$ is the result of splicing together the complexes

$$\mathbb{B}: 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_{l+1} \to F_l \to F_l' \to 0,$$

and

$$\mathbb{A}': 0 \to F_l' \xrightarrow{\varphi_l'} F_{l-1} \xrightarrow{\varphi_{l-1}} F_{l-2} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

at F_l'. (We admit the possibilities $l = n$, $\mathbb{B} = 0$, and $l = 0$, $\mathbb{A}' = 0$). It thus suffices to prove that $\mathbb{B} \otimes M$ and $\mathbb{A}' \otimes M$ are exact; Lemma 1 does this for $\mathbb{B} \otimes M$.

We complete the proof by an induction on the Krull dimension of R. If $\dim R = 0$, then $l = 0$, so $\mathbb{A}'$ is trivial, and we are done. Since the hypotheses of the theorem are not weakened by localizing, we may assume, by induction, that $\mathbb{A}' \otimes M$ becomes exact when it is localized at any *non*maximal prime ideal. Any nonvanishing homology groups must therefore have support consisting of the maximal ideal alone. In particular, they have depth 0. The next Lemma, which appears in [7, Lemma 1.8] thus finishes the proof when applied to the complex $\mathbb{A}' \otimes M: 0 \to F_l' \otimes M \to \cdots \to F_0 \otimes M$. We include a proof for completeness.

LEMMA 3. *Suppose R is a commutative noetherian ring, I an ideal, and*

$$\mathbb{C}: 0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0$$

a complex of R-modules with homology $H_k = H_k(\mathbb{C})$ such that for $k = 1,\ldots, n$,

(1) $\operatorname{depth}(I, M_k) \geqslant k$

(2) $\operatorname{depth}(I, H_k) = 0$

Then $\mathbb{C}$ is exact.

Proof. Write $B_k \subseteq C_k \subseteq M_k$ for the k-boundaries and k-cycles of the complex. Suppose $\mathbb{C}$ is not exact, and let $m \geqslant 1$ be the largest integer with $H_m \neq 0$, so that $B_m \neq C_m$, but

$$\mathbb{C}': 0 \to M_n \to \cdots \to M_{m+1} \to B_m \to 0$$

is exact.

It is easy to prove by induction on the length of C', that $\operatorname{depth}(I, B_m) \geqslant m + 1 \geqslant 2$; for, if $0 \to M_n \to M_{n-1} \to B_{n-2} \to 0$ is exact, with $\operatorname{depth}(I, M_k) \geqslant k$ for $k = n - 1, n$, then using the Ext characterization of depth, we see $\operatorname{depth}(I, B_{n-2}) \geqslant n - 1$. In the general case, we simply note that the sequence $\mathbb{C}'$ is the composite of the sequences

$$0 \to M_n \to M_{n-1} \to B_{n-2} \to 0$$

and

$$0 \to B_{n-2} \to M_{n-2} \to \cdots \to M_{m+1} \to B_m \to 0$$

both shorter than $\mathbb{C}'$. This establishes the fact that $\operatorname{depth}(I, B_m) \geqslant m + 1 \geqslant 2$.

Now since $C_m \subset M_m$ and $\operatorname{depth}(I, M_m) \geqslant m \geqslant 1$, we have $\operatorname{depth}(I, C_m) \geqslant 1$ also. Using the exact sequence

$$0 \to B_m \to C_m \to H_m \to 0$$

together with our information about depth(I, C_m) and depth(I, B_m), we deduce that depth(I, H_m) > 0 which is a contradiction since we have assumed that depth(I, H_m) $= 0$. Thus we have established the exactness of $\mathbb{C}$. ∎

3. Proof that the Exactness of $\mathbb{A} \otimes M$ implies (a) and (b)

We now suppose that $\mathbb{A} \otimes M$ is exact. We need to be able to localize without decreasing the numbers rank(φ_k, M). The next lemma enables us to do this by telling us of the existence of nonzerodivisors in each $I(\varphi_k, M)$.

Lemma 4. *Let R be a noetherian ring, $M \neq 0$ an R-module, and*

$$\mathbb{A}: 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

a complex of finitely generated free R-modules $F_k \neq 0$, such that $\mathbb{A} \otimes M$ is exact. Then for each $k = 1,\ldots, n$, $I(\varphi_k, M)$ contains a nonzerodivisor on M.

Remark. More is true; as in Bourbaki [1, III, p. 88, Cor.], if $F \to G$ is a map of finitely generated projective modules over a commutative ring R, M an R-module such that $F \otimes M \to G \otimes M$ is a monomorphism, then $(\wedge F) \otimes M \to (\wedge G) \otimes M$ is a monomorphism, where $\wedge F$ and $\wedge G$ denote the exterior algebras of F and G.

Proof of Lemma 4. If every element of $I(\varphi_n, M)$ were a zerodivisor on M, then there would be a submodule $M' \subset M$, $M' \neq 0$, such that $I(\varphi_n, M)\, M' = 0$. Since rank$(\varphi_n, M) \leqslant \operatorname{rank} F_n$, we have

$$\operatorname{rank}(\varphi_n, M') < \operatorname{rank}(\varphi_n, M) \leqslant \operatorname{rank} F_n .$$

If $I(\varphi_n, M')$ contains no nonzerodivisor on M', there is a submodule $M'' \subseteq M'$, $M'' \neq 0$, such that $I(\varphi_n, M')\, M'' = 0$. Thus

$$\operatorname{rank}(\varphi_n, M'') < \operatorname{rank}(\varphi_n, M') < \operatorname{rank} F_n .$$

Continuing in this way, we see that there is a submodule $L \subseteq M$, such that either $\operatorname{rank}(\varphi_n, L) = 0$, or $0 < \operatorname{rank}(\varphi_n, L) < \operatorname{rank} F_n$ and $I(\varphi_n, L)$ contains a nonzerodivisor on L. In either case, there is a commutative diagram

$$\begin{array}{ccc} F_n \otimes M & \xrightarrow{\varphi_n \otimes M} & F_{n-1} \otimes M \\ \uparrow & & \uparrow \\ F_n \otimes L & \xrightarrow{\varphi_n \otimes L} & F_{n-1} \otimes L \end{array}$$

where the top map and the two vertical maps are monomorphisms. Thus $\varphi_n \otimes L$ is a monomorphism. But if $\operatorname{rank}(\varphi_n, L) = 0$, $\varphi_n \otimes L = 0$, contradicting the assumption $L \neq 0$. Thus we may assume

$$0 \neq \operatorname{rank}(\varphi_n, L) < \operatorname{rank} F_n, \qquad \text{and} \qquad I(\varphi_n, L)$$

contains a nonzerodivisor r on L. Inverting r, we may assume $I(\varphi_n, L) = R$. But Lemma 2 then shows $\operatorname{rank}(\varphi_n, L) = \operatorname{rank} F_n$, another contradiction. Thus $I(\varphi_n, M)$ contains a nonzerodivisor on M.

To extend the result to the other φ_k, we begin by inverting the elements of R which are nonzerodivisors on M, and thus we may assume that all such elements are units in R. By factoring out the annihilator of M, we may further assume $\operatorname{rank}(\varphi_k, M) = \operatorname{rank}(\varphi_k)$, $I(\varphi_k, M) = I(\varphi_k)$ for all k.

Since we have already shown that $I(\varphi_n, M)$ contains a nonzerodivisor on M, and since we have assumed that the nonzerodivisors on M are units in R, we see that $I(\varphi_n) = R$ so that $\operatorname{coker} \varphi_n$ is projective. Denote this cokernel by F'_{n-1}. Actually, F'_{n-1} is not only projective, it is free. To see this, first observe that R is a semilocal ring (since every nonzerodivisor of M is a unit of R and hence every maximal ideal of R is in $\operatorname{ass}(M)$). Next, observe that at every localization of R,

$$\operatorname{rank} F'_{n-1} = \operatorname{rank} F_{n-1} - \operatorname{rank} \varphi_n$$

and the integer on the right is independent of the localization. Thus F'_{n-1} is a finitely generated projective module of constant rank over a semilocal ring and is therefore free. Write $\varphi'_{n-1} : F'_{n-1} \to F_{n-2}$ for the map induced by $\varphi_{n-1} : F_{n-1} \to F_{n-2}$. Evidently, $I(\varphi_{n-1}) = I(\varphi'_{n-1})$ and $\varphi'_{n-1} \otimes M$ is a monomorphism, so as in the case of φ_n, $I(\varphi'_{n-1})$ contains a nonzerodivisor. Induction completes the proof. ∎

Lemma 4 allows us to prove condition a) easily:

LEMMA 5. *With the hypothesis of Lemma* 4, *we have*

$$\operatorname{rank}(\varphi_{k+1}, M) + \operatorname{rank}(\varphi_k, M) = \operatorname{rank} F_k \qquad \textit{for all} \quad k = 1, \ldots, n.$$

Proof. We may invert all nonzerodivisors on M, and thus by Lemma 4, assume $I(\varphi_k) = R$ for all k. Lemma 2 now yields the result. ∎

All that remains is to prove condition (b). If for some k, $I(\varphi_k, M)$ does not contain an M-sequence of length k, then there is a prime $P \supseteq I(\varphi_k, M)$ such that $\operatorname{depth}(I(\varphi_k, M)_P, M_P) < k$ [6, Theorem 135]. Localizing at P, and factoring out the annihilator of M, we may assume that R is local with maximal ideal P, and M is faithful, so that $\operatorname{rank}(\varphi_k, M) = \operatorname{rank} \varphi_k$ and

$I(\varphi_k, M) = I(\varphi_k)$ for all k. If $I(\varphi_n)$ contains a unit, we may replace the complex A with the complex

$$\mathbb{A}'\colon 0 \to \operatorname{coker} \varphi_n \xrightarrow{\varphi'_{n-1}} F_{n-2} \xrightarrow{\varphi_{n-2}} F_{n-3} \to \cdots \to F_0 .$$

As in the proof of Lemma 3, we know that coker φ_n is free, that

$$\operatorname{rank} \varphi'_{n-1} = \operatorname{rank} \varphi_{n-1}, \quad \text{and that} \quad I(\varphi'_{n-1}) = I(\varphi_{n-1}).$$

We may continue to do this until we have a complex where the left hand map does not split. Of course the complex we obtain will still have at least k nonzero maps since we have arranged that $I(\varphi_k) \subsetneq P$. Thus the next lemma finishes the proof of the theorem.

Lemma 6. *Let R be a local, noetherian ring with maximal ideal P, $M \neq 0$ a finitely generated R-module, and*

$$\mathbb{A}\colon 0 \to F_k \xrightarrow{\varphi_k} F_{k-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

a complex of finitely generated free R-modules, such that $\varphi_m \otimes M \neq 0$ for $m = 1,\ldots, k$. Suppose that depth$(P, M) = l < k$ *and $\mathbb{A} \otimes M$ is exact. Then the map φ_k splits.*

Proof. We must show that $I(\varphi_k) = R$; suppose this is not so. Then $P \supseteq I(\varphi_k)$, which, by Lemma 4, contains a nonzerodivisor on M. Writing C_j for $\operatorname{Im}(\varphi_{j+1} \otimes M) = \ker(\varphi_j \otimes M)$, $j \geqslant 0$, we will show by induction on m that $\operatorname{depth}(P, C_{k-m-1}) = l - m$. Since $k - l - 1 \geqslant 0$, there will be a $C_{k-l-1} \subseteq F_{k-l-1} \otimes M$, $C_{k-l-1} \neq 0$, and $\operatorname{depth}(P, C_{k-l-1}) = 0$; this is ridiculous, since P contains a nonzerodivisor on M.

To show $\operatorname{depth}(P, C_{k-2}) = l - 1$, we examine the exact sequence $0 \to F_k \otimes M \to F_{k-1} \otimes M \to C_{k-2} \to 0$. The corresponding long exact sequence in $\operatorname{Ext}(R/P, -)$ gives

$$\cdots \to \operatorname{Ext}^{l-1}(R/P, F_{k-1} \otimes M) \to \operatorname{Ext}^{l-1}(R/P, C_{k-2})$$
$$\to \operatorname{Ext}^{l}(R/P, F_k \otimes M) \xrightarrow{\psi} \operatorname{Ext}^{l}(R/P, F_{k-1} \otimes M) \to \cdots.$$

Using the characterization of depth$(P, -)$ in terms of ext$(R/P, -)$ and the hypothesis on depth$(P, -)$ we see that $\operatorname{depth}(P, C_{k-2}) = l - 1$ unless the map we have labelled ψ is a monomorphism. But ψ is the map

$$\varphi_k \otimes \operatorname{Ext}^l(R/P, M).$$

The annihilator of $\operatorname{Ext}^l(R/P, M)$ obviously contains P, and by hypothesis $I(\varphi_k) \subsetneq P$. Thus

$$\operatorname{rank}(\varphi_k, \operatorname{Ext}^l(R/P, M)) < \operatorname{rank} F_k .$$

But by Lemma 5, this shows that $\varphi_k \otimes \mathrm{Ext}^l(R/P, M)$ is not a monomorphism. Thus $\mathrm{depth}(P, C_{k-2}) = l - 1$. For each $1 \leqslant n \leqslant k - 2$ there is an exact sequence

$$0 \to C_n \to F_n \otimes M \to C_{n-1} \to 0.$$

Starting with $n = k - 2$, we see inductively that

$$\mathrm{depth}(P, C_{k-m-1}) = l - m$$

as required. ∎

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