THREE CONJECTURES ABOUT MODULES OVER POLYNOMIAL RINGS

by

David Eisenbud

and

E. Graham Evans, Jr.

I) Introduction. In [E-E,1] we observed that several results on generating modules and ideals, and several results about algebraic K-theory, all depended on essentially the same arguments. In [E-E,2] we showed that one of these results - the theorem of Kronecker that every ideal in a noetherian d-dimensional ring can be generated, up to radical, by d+1 elements- can be strengthened in case the ground ring is a polynomial ring : we showed that if R is a noetherian d-dimensional ring of the form S[x], then every ideal of R can be generated, up to radical, by d elements.

This result made us ask whether the other results discussed in [E-E, 1] could also be strengthened in case the ground ring is a polynomial ring. A brief look at the literature produced a number of examples of results of this type. We were led to conjecture that improvements in most of the theorems in [E-E,1] should be possible in the polynomial ring case.

In this paper we present our conjectures, with some evidence for their validity. In section 2 we state the conjectures and list a number of the known theorems which are special cases. In section 3, we prove that all our conjectures hold for polynomial rings over semi-local rings of positive dimension. We also establish a theorem on the number of generators of a projective module of rank 1 which is another special case of our conjectures.

We recall from [E-E,1] some of the terminology and results that we will use:

Let R be a ring and let M be a finitely generated R-module. Then $\mu(R,M)$ is the minimal number of generators of M over R .

If p is a prime ideal of R , then dim(p) is the Krull dimension of R/p. If M is an R-module, $m \in M$, and p a prime ideal of R , then m is basic in M at p if $\mu(R_p, M_p) > \mu(R_p, (M/Rm)_p)$. m is a basic element if it is basic in M at p for every prime ideal p of R .

The height of a prime ideal p of R , ht(p) is the Krull dimension of R_p . $\mathcal{P}_t$ is the set of prime ideals, p , of R such that $ht(p) \leq t$. If p is a prime ideal of R , $\dim_t(p)$ is the length of the longest chain of elements of $\mathcal{P}_t$ which contain p .

For the reader's convenience, we state two of the results from [E-E,1] that we will use in section 3 . The first result concerns the existence of basic elements. We will state it only for the special case which we will use here.

Theorem A: [EE-1] : Let R be a noetherian ring, with $\dim R = d < \infty$. Let $M' \subseteq M$ be finitely generated R-modules, and suppose that for every prime ideal p of R , M' is (dim(p)+1)-fold basic in M at p . If $m_1,\ldots,m_u \in M'$ generate M' , and if $r \in R$ is given such that $(r,m_1) \in R \oplus M$ is basic, then there is a basic element of M of the form m_1+rm', where $m' \in \sum_{i=2}^{u} Rm_i$.

The other result is that basic elements in projective modules are unimodular:

Lemma 1 [E-E,1]. If R is a commutative ring, and P is a finitely generated projective R-module, then an element $m \in P$ is basic if and only if it generates a free direct summand of P .

Section 2. The Conjectures, and a Survey of Known Special Cases.

The conjectures we wish to consider are the following: Let S be a noetherian ring, and let $R = S[x]$ be the polynomial ring. Set $d = \dim R$.

1) If M is a finitely generated R-module such that $\mu(R_p, M_p) \geq d$ for every prime ideal p of R, then M has a basic element. In particular, if P is a projective R-module of rank d, then P has a free summand.

2) If P is a finitely generated projective R-module of rank $\geq d$ and if Q is a finitely generated projective R-module such that $Q \oplus P \cong Q \oplus P'$, then $P \cong P'$.

3) Let M be a finitely generated R-module, and let $\mathcal{P}$ be the set of primes of R such that $\dim(p) < d$. Set

$$n = \max_{p \in \mathcal{P}} (\mu(R_p, M_p) + \dim(p)).$$

Then M can be generated by n elements.

All three statements become true [E-E,1] if d is replaced by d+1.

It is easy to see that given their generality, these conjectures are the strongest possible. To see that this is so for conjecture 1, suppose that J is a maximal ideal of height d and let $M = J \oplus \ldots \oplus J$ (d-1 times). Then M cannot have a basic element, for if $(r_1, \ldots, r_{d-1})$ were a basic element, then J would be the radical of the ideal ΣRr_i contradicting Krull's principal ideal theorem.

To see that conjecture 2 fails if we replace d by $d-1$, let S be the coordinate ring of the real 2-sphere, $S = \mathbb{R}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 - 1)$, and let $\bar{P}$ be the cokernel of $S \xrightarrow{(x_1, x_2, x_3)} S^3$. Then $\bar{P}$ is projective, and $\bar{P} \oplus S$ is free, but it is known that $\bar{P}$ is not free. If we set $R = S[x]$ and $P = \bar{P} \otimes_S R$ then rank $P = 2 = \dim R - 1$ and $R \oplus P \cong R^3$. However, if $P \cong R^2$, then $\bar{P} \cong P/XP \cong S^2$ which is a contradiction.

As for conjecture 3, Murthy [Mur,2] has given an example of an unmixed ideal I of height 2, in a ring of the form $R = K[X,Y,Z]$ where K is a field, such that I can be generated locally by 2 elements but requires 3 generators globally.

Conjecture 2 implies that if $R = K[X_1, \ldots, X_d]$ with K a field, then every projective of rank $\geq$ d is free. This is a weak form of Serre's problem.

The following Corollary illustrates the application of conjecture 3:

Corollary to Conjecture 3 : Let $R = S[X]$ be a noetherian polynomial ring, and let I be an ideal of R. Suppose that I can be generated locally by g elements. Then I can be generated by

$$\max(d, g + \dim R/I)$$

elements.

The Forster-Swan Theorem [E-E,1] implies in the above situation that I can be generated by

$$\max(d+1, g + \dim R/I)$$

elements.

It would be tempting to formulate a stronger version of conjecture 2, to parallel [E-E,1,Thm.Aiib].But this stronger form is false, as is any form strong enough to imply the Stable Range Theorem for the ring $K[X_1,\ldots,X_n]$, where K is the field of real numbers. (An example in [Vas] shows that $d+1$ is best possible value in this case).

We do not know whether, for a ring R satisfying this hypothesis of the conjectures, $E(d,R)$ is transitive on unimodular elements (See [Bass-2]).

We will now ennumerate the special cases of our conjectures that we have found in the literature.

The best known of these special cases is Seshadri's theorem that if S is a principal ideal ring, then every projective $S[X]$-module is free. This has been generalized by Serre, [Ser], Bass [Bass,1] and Murthy [Mur,1] to the case in which S is any 1-dimensional ring with only finitely many non-regular maximal ideals; in this case the theorem says that any projective $S[X]$-module is the direct sum of a free module and an ideal. (The freeness of all projectives in case

S is a principal ideal domain follows immediately from this statement.) This is precisely the conclusion of our conjecture 1 applied to projective modules. Moreover, in this case, conjecture 2 follows from conjecture 1 , since it is always possible to cancel a projective module from an ideal [Kap, p.76]. The truth of conjecture 3 in this case is open.

Another situation for which the truth of a part of conjecture 1, is known is that in which R is a polynomial ring in an odd number of variables over a field. Bass [Bass, 3] , Corollary 4.3, proves that in this case every projective P , with rank P = dim R , has a free summand.

On the non-projective side, the proof we gave in [E-E,2] for a slightly weaker result, actually implies that if , in conjecture 1 , M is a direct sum of ideals of R , then M has a basic element. (The connection between the result in [E-E,2] and basic elements is described in [E-E,1, Cor. 7 .]

We now turn to conjecture 2 . Endo [Endo] discussed the problem of when every projective over S[X,Y] is free, where S is a one-dimensional semi-local domain. His results give cases in which conjecture 2 holds. Bass and Schanuel [B-S] prove our conjecture for R = S[X] where S is a polynomial ring over a semi-local principal ideal domain. Bass in [Bass, 2] proves that if R = S[X] , where S is a polynomial ring over a semi-local ring of positive dimension, and if d = dim R , then E(n,R) is transitive on unimodular rows if n > d . This enables him to prove conjecture 2 in case P is free.

As for conjecture 3 , the statement is immediate in case dim S=0; it follows, for instance from the structure theorem for modules over a euclidean ring. Endo [Endo] proved conjecture 3 for maximal ideals in a ring of the form $S_o[X_1,\dots,X_n]$, where S_o is a semilocal principal ideal domain. In [Ger], Geramita proves conjecture 3 for maximal ideals M of S[X] , where S is a Dedekind domain .

Davis and Geramita [D-G] prove it for maximal ideals over rings of the form $R = S_o[X_1,\ldots,X_n]$, where S_o is an arbitrary semilocal ring of positive dimension.

A particularly interesting special case of conjecture 3 is implied by a theorem of Murthy [Mur,2] , which shows that conjecture 3 holds if R is the ring of polynomials in 3 variables over a field, and M is any ideal of R .

Section 3 . 2 Special Cases of the Conjectures .

In this section we will establish all three of our conjectures in the case in which R has the form $R = S[X_1,\ldots,X_n]$, where S is a semi-local noetherian ring of positive dimension. This result includes a number of the known special cases of the conjectures which were mentioned in the previous section. We will also prove that conjecture 3 always holds for projective modules of rank 1 .

Theorem 1 . Let S be a semi-local ring with a noetherian spectrum of dimension > 0 , and let $R = S[X_1,\ldots,X_n]$. Set d = dim R . Then the three conjectures of section 2 hold for R .

Remarks. The hypothesis on S can be weakened to be that S has only finitely many prime ideals of maximal height. The only modification in the proof that would be necessary is the replacement of the Jacobson radical of S by the intersection of the primes of S of maximal height. A version of Theorem 1 using j-primes, etc., in the style of [E-E,1] may be proved just as we will prove Theorem 1 . Presumably, some non-commutative version of Theorem 1 , as in [E-E,1], is also true.

All three parts of Theorem 1 depend on the following lemma.

Lemma 2: Let R be a noetherian ring, and let I be an ideal of R . Let $K \subseteq M$ be finitely generated R-modules, and let t be an integer such that for every prime $p \in \mathcal{P}_t$, K is $(\dim_t(p)+1)$-fold basic in M at p . Suppose that $r \in R$ and $k \in K$ are elements such that

a) $(r,k) \in R \oplus M$ is basic , and

b) The image of k in M/IM is basic.

Then there exists an element $k' \in IK$ such that $k + rk'$ is basic in M at p for all $p \in \mathcal{P}_t$.

Proof of Theorem 1: Let J be the Jacobson radical of S , and set I = JR. The hypothesis on R and S shows that $\dim(R/I) < d$.

1) The second statement follows from the first because a basic element in a projective R-module generates a free summand [E-E, 1, Lemma 1].

To prove the first statement, we note that, for every prime p of R which contains I , we have

$$\mu((R/I)_{\bar{p}}, \ (M/IM)_{\bar{p}}) = \mu(R_p, M_p) \geq d ,$$

where we have written $\bar{p}$ for the reduction of p modulo I . Since dim (R/I) < d , there exists a basic element in M/IM by [E-E, 1]. Let $m \in M$ be an element which reduces to this basic element. Applying the Lemma , with K = M and t = d-1, to the basic element $(1,m) \in R \oplus M$, we see that there exists $m' \in IM$ such that $m + m'$ is basic in M at every prime ideal $p \in \mathcal{P}_{d-1}$.

We will show that $m + m'$ is basic at all primes of R . If q is a prime ideal of R with ht q = d , then $q \cap S$ is a maximal ideal of S . Thus $q \supseteq I$. On the other hand, $m + m'$ is basic modulo I since $m + m' \equiv m \pmod{I}$. This shows that $m + m'$ is basic at q, as required.

2.) We begin by making the familiar reduction to the case Q = R: there exists a projective module Q' such that $Q \oplus Q'$ is free, so it is enough to be able to cancel the rank 1 free summands of $Q \oplus Q'$ one at a time. Thus we may suppose $R \oplus P \cong R \oplus P'$. Let $f: R \oplus P' \to R \oplus P$ be the isomorphism. $(1,0) \in R \oplus P'$ is clearly basic, so if $f((1,0)) = (r,m_o) \in R \oplus P$, then (r,m_o) is basic. We will show that some automorphism a of $R \oplus P$ carries (r,m_o) to $(1,0)$. We will thus obtain a commutative diagram with exact rows of the following form:

$$\begin{array}{ccccccccc} 0 & \to & R & \to & R \oplus P' & \to & P' & \to & 0 \\ & & \| & & \downarrow af & & & & \\ 0 & \to & R & \to & R \oplus P & \to & P & \to & 0 \end{array} .$$

This shows that $P \cong P'$.

To construct a, we again use [E-E, 1 Theorem A] and the fact that $\dim (R/I) < d$ to show that there exists an element m_1 in P such that $\bar{m}_o + \bar{r}\bar{m}_1$ is basic in P/IP, where , we have written $\bar{\ }$ to denote reduction modulo I.

We can now apply the Lemma with $K = M$ and $t = d-1$ to the element $(r, m_o + rm_1) \in R \oplus P$. By the Lemma, there exists an element $m_2 \in IP$ such that $m_o + rm_1 + rm_2$ is basic at all primes $p \in \mathcal{P}_{d-1}$. As in part 1, above, it follows that the element $m = m_o + rm_1 + rm_2$ is basic in P.

The rest of the proof follows the pattern in [Bass, 2] and [E-E, 1,Cor.4]: If α denotes the map $R \to P$ carrying 1 to m_1+m_2, β denotes a map $P \to R$ carrying m to $1-r$ (such maps exist because, by Lemma 1, m generates a free summand of P), and γ denotes the map $R \to P$ taking 1 to $-m$, then we may take

$$a = \begin{pmatrix} 1 + \beta\alpha & \beta \\ \gamma+\alpha+\gamma\beta\alpha & 1 + \gamma\beta \end{pmatrix}$$

3) Unless $n < \max\limits_{\substack{\dim p = d \\ M_p \neq 0}} (d + \mu(R_p, M_p))$, this is the conclusion of the usual Forster-Swan theorem [E-E,1]. Therefore there must exist primes p of dimension d such that $M_p \neq 0$, and so, a fortiori, there exist primes p of dimension $< d-1$ such that $M_p \neq 0$. Thus $n \geq d$.

Now suppose $\mu(R,M) = t > n$. It follows that there is a short exact sequence of the form

$$(*) \qquad 0 \to K \to R^t \to M \to 0$$

Following the pattern of [E-E,1, Cor. 5], we will show that there exists an element $m \in K$ which is basic in R^t, so that R^t $R^t = Rm \oplus P$, by [E-E,1 , Lemma 1] , for some projective module P. We will then have rank $P = t-1 \geq d$, so by part 2), above, P is free of rank $t-1$. On the other hand, the epimorphism $R^t \to M$ induces an epimorphism $P \to M$, so M can be generated by $t-1$ elements. This contradicts our assumption that $\mu(R,M) = t$, proving the theorem.

It remains to show that K contains a basic element of R^t. From (*) and the assumption $t > n$, it follows that K is ($\dim p+1$)-fold basic in R^t at every prime p such that $\dim p < d$, and K is d-fold basic in R^t at primes p such that $\dim P = d$.

It is easy to see that if a prime ideal p contains the ideal I, then the image of K in R^t/IR^t is just as basic at p as is K in R^t. Thus [E-E, 1, Thm. A] implies that the image of K in R^t/IR^t contains a basic element $\bar{k}$ of R^t/IR^t. Let $k \in K$ be an element which reduces to $\bar{k}$ modulo IR^t. Applying the Lemma with $t = d-1$ to the basic element $(1,k) \in R \oplus R^t$, we see that there exists an element $k' \in IK$ such that $k + k'$ is basic at all prime ideals $p \in \mathcal{P}_{d-1}$. Since k is basic in R^t modulo I, it follows as in the proof of 1) that $k+k' = m$ is basic in R^t, as required.

<u>Sketch of Proof of Lemma 2</u> : The proof of this Lemma follows the pattern of the proof of Theorem A given in [E-E, 1] so closely that we will not give it in detail. Instead, we will remark on the points in the proof of Theorem A at which changes must be made. We assume that the reader has a copy of [E-E,1] before him.

The main difference between Lemma 2 of this paper, and Theorem A ii) b) of [E-E,1] with $A = R$, is that in Lemma 2, m_1 is assumed basic mod I , and we wish to produce elements a_i (in the notation of Theorem A!) which lie in I . (The appearance of the sets θ_t is an essentially trivial change). The elements a_i are actually obtained by a number of applications of Lemma 3 of [E-E,1]. In the notation of Lemma 3 , it suffices to prove that we may take $a_1 \in I$ (the other elements a_j in Lemma 3 don't matter). Turning to the proof of Lemma 3 , in section 5 of [E-E,1] , we see that a_1 is chosen so that, in the notation of [E-E,1],

$$m_1 + aa_1m_1$$

is basic in M at a certain finite list of primes $p_1,\ldots,p_v$. Suppose that m_1 is basic mod I, and that $p_1,\ldots,p_{v'}$ are the primes of this list that contain I . Then any choice of $a_1 \in I$ makes $m_1+aa_1m_1$ basic mod I, and therefore basic at each of the primes $p_1,\ldots,p_{v'}$. Thus it suffices to deal with the primes $p_{v'+1},\ldots,p_v$, which do <u>not</u> contain I . To do this, we first choose a_1 in the way that the proof of Lemma 3 instructs us- not necessarily in I. Then we pick $s \in I$ such that $s \notin \bigcup_{i=v'+1}^{v} P_i$; this is possible, since otherwise I would be contained in one of the primes in the union. We can now replace a_1 by $sa_1 \in I$ and continue with the proof as given in [E-E,1].

The next theorem covers a special case of conjecture 3). The proof is similar to that in [E-E,2].

<u>Theorem 2.</u> Let S be a noetherian ring, and let R = S[X] be the polynomial ring, and let P be a projective R-module of rank 1 . If dim R = d , then P can be generated by d elements.

<u>Proof</u>. We proceed by induction on d-1 = dim S . If dim S = 0 , then S is artinian. Let N be its nilpotent radical. By Nakayama's lemma,

$$\mu(R,P) = \mu(R/NR, \quad P/NP).$$

Thus we may assume N = 0, so that S is a direct product of fields. But in this case R is a direct product of principal ideal domains, so P is cyclic.

Now suppose that $\dim S > 0$. Let $\mathcal{U} \subseteq S$ be the multiplicative subset which is the complement of the union of the minimal primes of S .

Then $S_{\mathcal{U}}$ has dimension 0 , so by induction, $P_{\mathcal{U}}$ can be generrated over $S_{\mathcal{U}}[X] = R$ by one element $p_1 \in P$. It follows that there exists an element $u \in \mathcal{U}$ such that

$$uP \subseteq Rp_1 \tag{3}$$

Since u is not in any of the minimal primes of S , $\dim S/(u) < \dim S$. Thus, by induction, the rank 1 projective $R/(u)$-module P/uP can be generated by $d-1$ element $\overline{p}_2,\ldots,\overline{p}_d$, that is ,

$$P/uP = \sum_{i=2}^{d} R\overline{p}_i \quad . \tag{4}$$

We claim that if $p_2,\ldots,p_d$ are any elements of P which reduce modulo (u) to $\overline{p}_2,\ldots,\overline{p}_d$, then

$$P = \sum_{i=1}^{d} Rp_i$$

It is enough to prove that this holds after localizing at an arbitrary prime ideal q of R . If $u \notin q$ then

$$P_q = \Big(\sum_{i=1}^{d} Rp_i \Big)_q \tag{5}$$

follows immediately from (3). If, on the other hand, $u \in q$, then (5) follows from Nakayama's lemma and (4). This concludes the proof.

REFERENCES

Bass, H., [Bass, 1] "Torsion Free and Projective Modules," Trans. A.M.S. 102 (1962), 319-327.

[Bass, 2] "K-Theory and Stable Algebra " Publ. Math. IHES no. 22 (1964), 5-60.

[Bass, 3] "Modules Which Support Nonsingular Forms," J. Alg. 13 (1969), 246-252.

Bass, H., and Schanuel, S. [B-S] "The Homotopy Theory of Projective Modules," Bull. A.M.S. 68 (1962), 425-428.

Davis, E., and Geramita, A., [D-G] "Maximal Ideals in Polynomial Rings." these Proceedings.

Eisenbud, D., and Evans, E. [E-E,1] "Generating Modules Efficiently: Theorems from Algebraic K-Theory." J. Alg., To appear.

[E-E, 2] "Every Algebraic Set in n-space is the Intersection of n Hypersurfaces." To appear.

Endo, S. [Endo] "Projective Modules Over Polynomial Rings." J. Math Soc. Japan 15 (1963), 339-352.

Geramita, A. [Ger], "Maximal Ideals in Polynomial Rings ," Queen's University Mathematical Preprint no. 1971-56.

Kaplansky, I., [Kap] Infinite Abelian Groups, Revised edition, The University of Michigan, Ann Arbor, (1969).

Kronecker, L., [Kro] "Grundzüge eine arithmetischen Theorie der algebraischen Grossen," J. Reine. Angew. Math. 92(1882), 1-123.

Murthy, P. [Mur,1] "Projective Modules Over a Class of Polynomial Rings," Math. Z. 88 (1965), 184-189.

[Mur, 2] "Generators for Certain Ideals in Regular Rings of Dimension three" To appear.

Serre, J.-P. [Ser] "Sur les Modules Projectifs," Sem. Dubriel-Pisot. t. 14 , 1960-1961, no. 2 .

Seshadri, C. [Ses] "Triviality of Vector Bundles over the Affine Space K^2," Proc. Nat. Acad. Sci. 44(1958), 456-458.

Vaserstein, L. [Vas] "Stable Rank of Rings and Dimensionality of Topological Spaces," Functional Analysis and its Applications 5 , (1971), 102-110.

Brandeis University
Waltham, Massachusetts

M.I.T.
Cambridge, Massachusetts