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Serial Rings

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A module is called *uniserial* if it has a unique composition series of finite length. A ring (always with 1) is called *serial* if its right and left free modules are direct sums of uniserial modules.

Nakayama, who called these rings generalized uniserial rings, proved [21, Theorem 17] that every finitely generated module over a serial ring is a direct sum of uniserial modules. In section one we give a short conceptual proof of this result, strengthening it to arbitrary modules (Theorem 1.2). As a byproduct of the proof, we obtain a condition for a projective module over a serial ring to be injective (Theorem 1.4).

More can be said about the structure of modules over a serial ring. In section two we show that the endomorphism ring of a projective module over a serial ring is a local serial ring (Corollary 2.2), and that the composition series of any uniserial module over a serial ring is periodic in a strong sense (Theorem 2.3). The section concludes with the theorem that any two simple modules over an indecomposable serial ring have the same endomorphism ring (Theorem 2.4).

Serial rings occur naturally in several contexts. It has long been known that every proper factor ring of a (commutative) Dedekind domain is an artinian principal ideal ring. (Commutative or not, any artinian principal ideal ring is serial.) Although it is also true that factor rings of Dedekind prime rings are artinian principal ideal rings [23, Theorem 3.5], this fails for hereditary noetherian prime rings in general (see [8, Sec. 4]). However, we prove in section three that any artinian factor ring of an hereditary ring with a flat injective envelope is serial (Theorem 3.1). In particular, every proper factor ring of an hereditary noetherian prime ring is serial.

Are Dedekind prime rings precisely the hereditary noetherian prime rings whose factor rings are principal ideal rings? The answer is "yes" under mild

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additional hypotheses (Theorem 3.3). The general question remains open, though Faith [10] and Levy [16] have completely characterized commutative rings whose proper factor rings are artinian principal ideal rings, and Zaks [26] has some noncommutative results.

In section four we show that for a finite dimensional algebra, the serial property is stable under changes of base field. More precisely, if A is a finite dimensional algebra such that $A/\text{Rad } A$ is separable, then A is serial if and only if it becomes serial when tensored with any algebraically closed field.

Serial algebras occur as the group algebras in characteristic p of certain finite groups. This class of groups includes, by a result of Srinivasan [24, Theorem 3], all p-solvable groups with cyclic sylow p-subgroups. Janusz [14, Corollary 7.5] gives a necessary and sufficient condition for a group algebra to be serial over a splitting field of characteristic p. The theorem we prove in section four has the consequence that Janusz's condition determines all serial group algebras.

Since the literature on serial rings is rather widely scattered, we have tried to make our bibliography complete enough so that it and the bibliographies of the papers listed would form a relatively complete guide.

For the reader's convenience, we have collected some miscellaneous notation and terminology which we will use throughout this paper.

If a ring A has (Jacobson) radical N, then the Loewy length of an A-module M is the smallest integer k, if one exists, such that $N^kM = 0$. Note that M is uniserial if and only if $M \supset NM \supset N^2M \cdots$ is a composition series for M.

For any module M, soc M denotes the sum of the simple submodules of M. If A is semiprimary, then a direct summand of the free left A-module of rank one is called a dominant left summand of A if it has maximal Loewy length.

An indecomposable ring is one which is not the (two-sided) direct sum of ideals. A local ring is one in which the nonunits form an ideal. We will always abbreviate quasi-Frobenius to QF.

For unexplained terminology concerning homomorphisms and modules, see MacLane [17].

The reader should be warned that we have thought of endomorphisms as acting opposite ring elements whenever this was necessary to prevent the appearance of opposite rings. With this exception, "module" is generally understood as "left module".

Note. An example due to J. C. Robson shows that an hereditary noetherian prime ring whose proper factor rings are all artinian principal ideal rings need *not* be Dedekind.

1. Nakayama's Theorem Revisited

Lemma 1.1 gives a characterization of rings whose left modules are direct sums of uniserial modules, and prepares the way for our proof of Nakayama's Theorem (Theorem 1.2) which is the basic result on serial rings. Theorem 1.4 completes the proof by telling which projective modules over a serial ring are injective.

Proposition 1.1. *Let A be a left artinian ring with radical N. Then every left A-module is a direct sum of uniserial A-modules if and only if the dominant left summands of A/N^k are A/N^k-injective for every k.*

Proof. $\Rightarrow$: The hypothesis remains valid for any homomorphic image of A, so it suffices to show that every dominant left summand X of A is injective. Since the injective envelope of X must be uniserial (and therefore a homomorphic image of an indecomposable summand of A), this is clear by the maximality of the length of X.

$\Leftarrow$: We first prove that every indecomposable left summand X of A is uniserial. This will follow if $N^{k-1}X/N^kX$ is simple or zero for every k. If $N^{k-1}X/N^kX \neq 0$, then X/N^kX is a dominant indecomposable summand of A/N^k and thus is an indecomposable A/N^k-injective. Consequently $\operatorname{soc}(X/N^kX)$ is simple, so $N^{k-1}X/N^kX \subseteq \operatorname{soc}(X/N^kX)$ is also simple.

Next we show that every nonzero left A-module has a uniserial summand. Since A is a sum of uniserial modules, every left module M is generated by its uniserial submodules. Suppose $X \subseteq M$ is a uniserial submodule of maximal length; say the length of X is k. Necessarily, $N^kM = 0$, so the inclusion $X \to M$ is a map of A/N^k-modules. Since X is isomorphic to a dominant summand of A/N^k, X is A/N^k-injective. Thus X is a summand of M.

What we have just proved shows that every finitely generated left A-module is a direct sum of uniserial modules. To obtain the general case, we use the notion of purity: details may be found in [17, pp. 367–375] and in [25]. A submodule B of the left A-module C is called *pure* if for every right A-module D, the induced map $D \otimes_A B \to D \otimes_A C$ is a monomorphism. If B is a pure submodule of C and C/B is finitely presented, then $C \cong B \oplus (C/B)$.

Returning to the proof at hand, we note that since direct summands are always pure, and since the filtered union of pure submodules is again a pure submodule, we may choose, in any A-module M, a maximal pure submodule of the form $\coprod X_i = M'$, where the X_i are uniserial modules. If $M/M' \neq 0$, it has a uniserial direct summand X. Let P be the preimage of X in M; M' is easily seen to be a pure submodule of P, so $P \cong M' \oplus X$, a direct sum of uniserial modules. However, P is a pure submodule of M, contradicting the maximality of M'. Thus $M' = M$ is a direct sum of uniserial modules. //

In [12, Theorem 3.6], Fuller proves that a condition similar to that of Lemma 1.1 is equivalent to the condition that A be serial. However his proof of this involves a quotation of Nakayama's Theorem.

THEOREM 1.2 (Nakayama, [21, Thm. 17]). *Let A be a serial ring. Then every A-module is a direct sum of uniserial modules.*

Proof. Let $N = \operatorname{Rad} A$. Since A/N^k is serial for every k, it suffices, by Lemma 1.1, to prove that the dominant summands of A are injective. The next theorem does this and more. To prove it, we will make use of the following simple result of Auslander [3, Prop. 10]:

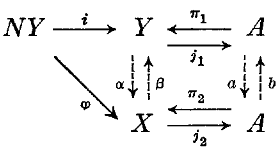
LEMMA 1.3. *Let A be an artinian ring, and let X be an A-module. Suppose* $\operatorname{Ext}_A{}^1 (S, X) = 0$ *for every simple module S. Then X is injective.* //

THEOREM 1.4. *Let A be a serial ring with radical N, and let X be an indecomposable summand of A. X is injective if and only if, for every indecomposable summand Y of A, $X \not\cong NY$.*

Proof. $\Rightarrow$: This is clear, since Y is assumed indecomposable.

$\Leftarrow$: By Lemma 1.3, it is enough to show that $\operatorname{Ext}_A{}^1 (Y/NY, X) = 0$ for every indecomposable summand Y of A, or, equivalently, that every map $\varphi : NY \to X$ extends to a map $Y \to X$.

Consider the diagram



where i, j_1 and j_2 are the natural inclusions, and π_1 and π_2 the projections. We wish to construct α making the left-hand triangle commute.

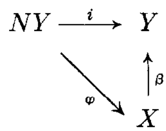
Since NY is uniserial, we can find a primitive idempotent e of A such that there is an epimorphism $Ae \twoheadrightarrow NY$. This induces a monomorphism

$$\operatorname{Hom}_A(NY, A) \rightarrowtail \operatorname{Hom}_A(Ae, A) \cong eA,$$

and eA is uniserial as a right A-module, since it is an indecomposable right summand of A. Consequently, one of $j_1 i$ and $j_2\varphi$, viewed as elements of eA, is a multiple of the other.

Suppose $j_2\varphi$ is a multiple of $j_1 i$, so that we can find an $a : A \to A$ such that $j_2\varphi = aj_1i$. Setting $\alpha = \pi_2aj_1$, we see that $\alpha i = \pi_2aj_1i = \pi_2j_2\varphi = \varphi$, as required.

If on the other hand, $j_1i = bj_2\varphi$, for $b : A \to A$, we set $\beta = \pi_1bj_2$. As before, $\beta\varphi = \pi_1bj_2\varphi = \pi_1j_1i = i$, so the diagram



commutes.

We need only show that β is an isomorphism. For then, setting $\alpha = \beta^{-1}$, we see that $\alpha i = \varphi$ as required.

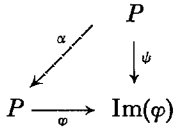
Certainly, $\mathrm{Im}(\beta) \supseteq NY$. If $\mathrm{Im}(\beta) = NY$, then $1_{NY} = \beta\varphi$, so that NY is a summand of X. Since X is indecomposable, $NY \simeq X$, contradicting our hypothesis. Hence $\mathrm{Im}\,\beta \supsetneqq NY$, and thus β is an epimorphism. Since Y is projective and X is indecomposable, β is an isomorphism, as required. This concludes the proof. //

2. Simple and Uniserial Modules over Serial Rings

We first examine the structure of a uniserial module over a serial ring A more closely. Any uniserial A-module is a homomorphic image of an indecomposable summand of A, so it is enough to study the projective uniserial modules.

Lemma 2.1. *Let A be an arbitrary ring. Suppose that X is a uniserial left A-module and P a projective left A-module. Then* $\mathrm{Hom}_A(P, X)$ *is a uniserial right* $\mathrm{End}_A(P)$-module. *Its* $\mathrm{End}(P)$-*submodules correspond to the submodules of X which are homomorphic images of P.*

Proof. Suppose $\varphi, \psi : P \to X$. It suffices to show that one of them is an $\mathrm{End}_A P$-multiple of the other. Now one of $\mathrm{Im}(\varphi)$ and $\mathrm{Im}(\psi)$ contains the other, say $\mathrm{Im}(\varphi) \supseteq \mathrm{Im}(\psi)$. By the projectivity of P, there is $\alpha : P \to P$ making the diagram



commutative.

Thus $\varphi\alpha = \psi$, proving the lemma. //

Corollary 2.2. *Let A be a serial ring, and let X be an indecomposable projective A-module. Then* $\mathrm{End}(X)$ *is a local serial ring.*

Proof. By Lemma 2.1, End(X) is uniserial as a right module over itself. Let $N = \text{Rad}(A)$, and let k be the Loewy length of X. By Theorem 1.4, X is injective over A/N^k. The dual of Lemma 2.1 shows that End(X) is also uniserial as a left module over itself.

THEOREM 2.3 (Periodicity Theorem). *Let A be a serial ring with radical N, and let X be any uniserial left A-module. Let the sequence of composition factors of X be $X_0 = X/NX$, $X_1 = NX/N^2X, \ldots$. Suppose $X_h \cong X_0$, and let $h \neq 0$ be the smallest such integer. Then $X_\ell \cong X_m$ if and only if $\ell \equiv m \pmod h$. If there is no h such that $X_h \cong X_0$, then $X_\ell \cong X_m$ implies $\ell = m$.*

Proof. Since X is the homomorphic image of a uniserial projective module, we may assume X is projective. Thus if $X_0 \cong X_h$, the map $X \to X_0 \cong X_h = N^hX/N^{h+1}X$ lifts to a map $X \to N^hX$ which sends $N^\ell X$ onto $N^{\ell+h}X$, inducing $X_\ell \cong X_{\ell+h}$ for every $\ell \geqslant 0$.

To finish the proof of the theorem, it now suffices to prove that if X_ℓ and X_m are composition factors of X/N^hX, then $X_\ell \cong X_m$ implies $\ell = m$. That is, it suffices to prove the last statement of the theorem. Thus we assume $X_\ell \ncong X_0$ for any $\ell \neq 0$, so $\text{End}_A(X)$ is a division ring.

Because X is projective and indecomposable, we may write $X \cong Ae$, where e is a primitive idempotent of A. Suppose $X_\ell \cong X_m \neq 0$ and let f be a primitive idempotent of A such that $Af/Nf \cong X_\ell$. Then

$$\text{Hom}_A(Af, Ae) \cong fAe \cong \text{Hom}_A(eA, fA)$$

as $fAf - eAe$ bimodules. By Lemma 2.1, fAe is a uniserial fAf module and a uniserial eAe module. Since the elements of eAe act as fAf-homomorphisms on fAe and vice versa, every one-sided submodule of fAe is two-sided. Thus fAe has the same length as a right and as a left module. Since fAe is uniserial and $eAe \cong \text{End}(X)$ is a division ring, this length must be one. Using Lemma 2.1 again, we see that only one submodule of Ae is a homomorphic image of Af. This submodule must be $N^\ell X$ *and* N^mX. Hence $\ell = m$. //

Remark. It is possible to give a slightly faster proof of Theorem 2.3 as follows: Use Theorem 1.4 to reduce to the case where X is both projective and injective. Then use Corollary 2.2, Lemma 2.1, and the categorical dual of Lemma 2.1 to finish the proof.

Though an indecomposable serial ring may have many isomorphism classes of simple modules, the next theorem shows that its simple modules must all have the same endomorphism ring. We will exploit this fact in the proof of Theorem 4.1.

THEOREM 2.4. *Let A be an indecomposable serial ring, and let S and T be simple A-modules. Then* $\text{End}_A(S) \cong \text{End}_A(T)$.

We will prove this as a consequence of:

LEMMA 2.5. *Let* $S \not\cong T$ *be simple modules over any ring, and suppose* $S \rightarrowtail X \twoheadrightarrow T$ *is exact but not split, with* X *projective and injective. Then* $\operatorname{End}(S) \cong \operatorname{End}(X) \cong \operatorname{End}(T)$.

Proof. We prove only the first isomorphism, the proof of the second being dual. Of course, it suffices to show that any $\varphi \in \operatorname{End}(S)$ extends uniquely to a $\varphi' \in \operatorname{End}(X)$, since then the map $\varphi \mapsto \varphi'$ is patently an isomorphism of rings. By the injectivity of X, φ may be extended to an element of $\operatorname{End}(X)$; suppose φ' and φ'' are two extensions. Then $(\varphi' - \varphi'')\, S = 0$, so $\operatorname{Im}(\varphi' - \varphi'') \subseteq \operatorname{soc} X = S$. Thus $\varphi' - \varphi''$ factors as in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi' - \varphi''} & X \\ \downarrow & & \uparrow \\ T \cong X/S & \longrightarrow & S \end{array} \quad .$$

Since $T \not\cong S$, we must have $\varphi' - \varphi'' = 0$. //

Proof of Theorem 2.4. Set $\operatorname{Rad} A = N$, and let $\{X_1, \ldots, X_n\}$ be a set of representatives of the isomorphism classes of indecomposable projective A-modules. By [15, Theorem 5] we may assume that the X_i are arranged so that for $i = 2, \ldots, n$, $X_{i-1}/NX_{i-1} \cong NX_i/N^2X_i$. Using Theorem 1.4, we see that for $i = 2, \ldots, n$, X_i/N^2X_i is projective and injective as an A/N^2-module. Any isomorphism $X_i/NX_i \cong X_j/NX_j$ would lift to an isomorphism $X_i \cong X_j$ by the A-projectivity of X_i. Thus for $i \neq j$, $X_i/NX_i \not\cong X_j/NX_j$. We can now apply Lemma 2.5 to the exact sequences

$$X_{i-1}/NX_{i-1} \rightarrowtail X_i/N^2X_i \twoheadrightarrow X_i/NX_i,$$

for $i = 2, \ldots, n$, and this finishes the proof. //

Remarks. 1) Theorem 2.4 says, in effect, that if A is an indecomposable serial ring, then $A/\operatorname{Rad} A$ is a product of full matrix rings over the same division ring.

2) In [1, section 2], Amdal and Ringdal state without proof that if S and T are simple modules over a serial ring, and if $\operatorname{Ext}^1(S, T) \neq 0$, then $\operatorname{Ext}^1(S, T)$ is one dimensional over $\operatorname{End}(S)$ and over $\operatorname{End}(T)$. The proof of Lemma 2.5 may be construed as proving this fact, from which it trivially follows that $\operatorname{End}(S)$ and $\operatorname{End}(T)$ are isomorphic. The application of Kuppisch's Theorem in our proof of Theorem 2.4 simply shows that one can go from one simple module to all the others in this way.

3) Of course an indecomposable serial ring may have many non-isomorphic simple modules. In fact, all its simple modules are isomorphic if and only if it is an artinian principal ideal ring; see, for example, [7].

3. Factor Rings of Hereditary Rings

THEOREM 3.1. *Let R be a left hereditary ring with a flat injective envelope. Then every left artinian factor ring of R is serial.*

In [9, Theorem 3.3] it was shown, by a direct assault, that factor rings of an hereditary noetherian prime ring with enough invertible ideals are all serial. Using Theorem 3.1, we can strengthen this result to include all hereditary noetherian prime rings.

COROLLARY 3.2. *Every proper factor ring of an hereditary noetherian prime ring is serial.*

Proof. By [8, Theorem 1.3], every factor ring of an hereditary noetherian prime ring R is artinian. Moreover, the injective envelope of R as a left R-module is easily seen to be the quotient ring Q of R. Since R is a right order in Q, we have $Q = \varinjlim_{a \text{ regular in } R} Ra^{-1}$. Since the $\varinjlim$ is filtered, Q is flat as a left R-module. //

Proof of Theorem 3.1. Let $A = R/I$ be left artinian. Since any statement we prove about A will apply equally to each factor ring of A, it suffices, by Lemma 1.1 and [12, Theorem 5.4], to prove that any dominant left summand of A is injective.

Let E be the left R-injective envelope of R. We start by calculating the left A-injective envelope of A in terms of E. Since R is left hereditary, E/I is R-injective. Let $F = \{x \in E \mid Ix \subseteq I\}$. Clearly F is a left R-submodule of E and is therefore flat. Set $F/I = E'$ and observe that E' is a left A-module containing A. E' is an injective A-module since $E' \cong \mathrm{Hom}_R(A, E/I) \cong \{x \in E/I \mid IX = 0\}$.

On the other hand, we claim that E' is a projective A-module. By [3, Corollary 8] it suffices to show that E' is a flat A-module. Note that $I = IF$ since $1 \in F$; hence $E' = F/IF \cong A \otimes_R F$. But F is a flat R-module, and therefore E' is a flat A-module.

Since E' is injective and projective, we may write $E' = \coprod X_i$, where the X_i are injective indecomposable left summands of A. Let X be a dominant left summand of A. We will show that X is isomorphic to one of the X_i. For every i, we have a map $\varphi_i : X \to X_i$ given by

$$\varphi_i = \left(X \xrightarrow{\text{incl.}} A \rightarrowtail E' = \coprod X_i \xrightarrow{\text{proj.}} X_i\right).$$

Since $X \rightarrowtail A \rightarrowtail E'$ is a monomorphism, we can choose an i so that $\operatorname{Ker} \varphi_i \not\supseteq \operatorname{soc} X$, and thus Loewy length(Im φ_i) = Loewy length(X). Since X is dominant, Loewy length(X_i) $\leqslant$ Loewy length(X), so φ_i is necessarily onto. Since X_i is projective, φ_i splits. Since X is indecomposable, φ_i is an isomorphism.

We have now shown that every dominant left summand of A (or, a fortiori, of any factor ring of A) is injective. Thus A is serial. //

Recall that an hereditary noetherian prime ring R is said to have enough invertible ideals if every ideal of R contains an invertible ideal. (For this and other notions concerning hereditary noetherian prime rings, see [9].)

PROPOSITION 3.3. *Let R be an hereditary noetherian prime ring with enough invertible ideals, and suppose that every factor ring of R is an artinian principal ideal ring. Then R is a Dedekind prime ring.*

Proof. Suppose R is not Dedekind, that is, suppose R has an idempotent maximal ideal I. Take J maximal among the invertible ideals contained in I; then $J \neq IJ$, and by [9, Theorem 2.6], $J/IJ = \operatorname{Rad}(R/IJ)$. Set $A = R/IJ$, $N = J/IJ$. By hypothesis A is an artinian principal ideal ring, so A is QF [22, Lemma 2]. Since J is invertible, the left annihilator of J/IJ is $I/IJ = \operatorname{soc} A$. We have

$$I/IJ \cong \coprod e_i A \oplus \coprod f_j N,$$

where the e_i and f_j are primitive idempotents of A such that the length of $e_i A$ is 1 and the length of $f_j A$ is 2. Since $N \neq 0$, the set of f_j's is not empty. But $I/IJ = (I/IJ)^2$ implies that some $e_i A \cong f_j N$, which contradicts the fact that A is QF. //

4. A STABILITY PROPERTY OF SERIAL ALGEBRAS

For results and definitions concerning splitting fields and separable algebras, see [6, Ch. 10, and 4].

THEOREM 4.1. *Let A be a finite dimensional algebra over the field k, and let K be any algebraically closed field containing k. Suppose $A/\operatorname{Rad} A$ is separable over k. Then A is serial if and only if $K \otimes_k A$ is serial.*

To prove 4.1 we will use the characterization of serial rings stated in [1, section 2]:

CRITERION 4.2. An artinian ring A is serial if and only if it satisfies the following condition and its dual: For every simple A-module S there is (up to isomorphism) at most one simple A module T such that $\operatorname{Ext}_A^1(S, T) \neq 0$. For such a T, $\operatorname{Ext}_A^1(S, T)$ is one-dimensional over $\operatorname{End}_A(S)$.

Proof of Theorem 4.1. $\Rightarrow$: Suppose that A is serial; of course we may assume that A is indecomposable. Let the endomorphism ring of a simple A-module be D (by Theorem 2.4, D is independent of the choice of simple module.) Suppose $[D : \text{Center}(D)] = n^2$, $[\text{Center}(D) : k] = p$.

Let S be any simple A-module. Since $A/\text{Rad}\, A$ is separable, $K \otimes S$ is semisimple, and we may write $K \otimes S = n \cdot \coprod_{i=1}^{p} S_i$, where the S_i are pairwise nonisomorphic simple $K \otimes A$ modules. Moreover, every simple $K \otimes A$ module is isomorphic to a module of the form S_i for some simple A-module S. Of course if T is a simple A-module with $T \not\cong S$, then $S_i \not\cong T_j$ for any i, j.

We proceed to verify the conditions of Criterion 4.2 for $K \otimes A$. Suppose $\text{Ext}^1_{K\otimes A}(S_i, T_j) \neq 0$ and $\text{Ext}^1_{K\otimes A}(S_i, U_k) \neq 0$ for S, T, U simple A-modules. Since

$$K \otimes \text{Ext}_A{}^1(S, T) = \text{Ext}^1_{K\otimes A}(K \otimes S, K \otimes T), \tag{*}$$

it follows that $\text{Ext}_A{}^1(S, T) \neq 0$ and $\text{Ext}_A{}^1(S, U) \neq 0$, whence $T \cong U$ by the criterion, applied to A.

By virtue of (*) and the fact that A is serial, $\text{Ext}^1_{K\otimes A}(K \otimes S, K \otimes T)$ is free of rank one over

$$K \otimes \text{End}_A S = \text{End}_{K\otimes A}(K \otimes S) = \prod_{i=1}^{p} M_n(\text{End}_{K\otimes A}(S_i))$$

$$= \prod_{i=1}^{p} M_n(K),$$

where $M_n(\text{End}_{K\otimes A}(S_i))$ is the $n \times n$ matrix ring over $\text{End}_{K\otimes A}(S_i)$.

Splitting this free module into summands corresponding to the matrix rings in the product, we see that

$$\coprod_{n \text{ copies}} \text{Ext}^1_{K\otimes A}(S_i, K \otimes T) = n^2 \cdot \coprod_{j=1}^{p} \text{Ext}^1_{K\otimes A}(S_i, T_j)$$

is free of rank one over $M_n(K)$, and thus has K-dimension n^2. This shows that $\text{Ext}^1_{K\otimes A}(S_i, T_j)$ is nonzero for exactly one value of j, and is one dimensional over K for that value, as required.

Since the dual argument yields the dual result, we have shown that Criterion 4.2 is satisfied for $K \otimes A$, so $K \otimes A$ is serial.

$\Leftarrow$: Now suppose $K \otimes A$ is serial. Fix a simple A-module S. As in the first part of the proof, we may write $K \otimes S = n \cdot \coprod_{i=1}^{p} S_i$, where the S_i are pairwise nonisomorphic simple $K \otimes A$-modules.

Now suppose U and T are simple A-modules with both $\text{Ext}_A{}^1(S, T) \neq 0$ and $\text{Ext}_A{}^1(S, U) \neq 0$. Both $\text{Ext}^1_{K\otimes A}(K \otimes S, K \otimes T)$ and $\text{Ext}^1_{K\otimes A}(K \otimes S,$

$K \otimes U$) are free over $\text{End}_{K\otimes A}(K \otimes S) = \prod_{i=1}^{p} M_n(\text{End}_{K\otimes A}(S_i))$, and thus for each S_i, $\text{Ext}^1_{K\otimes A}(S_i, K \otimes T)$ and $\text{Ext}^1_{K\otimes A}(S_i, K \otimes U)$ are nonzero. Since $K \otimes A$ is serial, $K \otimes T$ and $K \otimes U$ must have a simple factor in common, so $U \simeq T$.

We now need only show $\text{Ext}_A{}^1(S, T)$ is one dimensional over $\text{End}_A(S)$. Writing $K \otimes T = m \cdot \coprod_{j=1}^{q} T_j$, where the T_j are pairwise nonisomorphic simple $K \otimes A$-modules, we see that this amounts to showing that for each i, the free $M_n(\text{End}_{K\otimes A}(S_i)) = M_n(K)$-module

$$\coprod_{n \text{ copies}} \text{Ext}^1_{K\otimes A}\left(S_i, m \cdot \coprod_{j=1}^{q} T_j\right) = mn \cdot \coprod_{j=1}^{q} \text{Ext}^1_{K\otimes A}(S_i, T_j) \qquad (**)$$

has rank one. Since $K \otimes A$ is serial, we see from Criterion 4.2 that $\coprod_{j=1}^{q} \text{Ext}^1_{K\otimes A}(S_i, T_j)$ is one dimensional over $\text{End}_{K\otimes A}(S_i) = K$. Thus the module in (**) has dimension mn over K. Since it is free over $M_n(K)$, we must have $n^2 \mid mn$, that is, $n \mid m$. Using the dual part of the Criterion 4.2, and arguing on T rather than S, we see that $m \mid n$. Thus $m = n$.

Since the module in (**) is free over $M_n(K)$ and has K-dimension $n^2 = mn$, it must be free of rank one over M_nK. This shows that $\text{Ext}_A{}^1(S, T)$ is one dimensional over $\text{End}_A(S)$. A dual argument now establishes the dual part of the criterion for A, so A is serial. //

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