Math. Ann. 185, 247—249 (1970)

Subrings of Artinian and Noetherian Rings

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1. Main Results

It is well known that if $R \subset S$ are rings (rings in this paper have units but need not be commutative) such that S is finitely generated as a left R-module, then S is Noetherian or Artinian if R is. The converses of these statements are false [1; Example 1.0]. In this paper we give a short homological proof that the converse statements do hold under slightly stronger hypotheses:

Theorem 1. *Let $R \subset S$ be rings such that S is finitely generated as an R-module by elements which centralize R.*

Then:

a) *If S is left Noetherian, then R is left Noetherian.*

b) *If S is left Artinian, then R is left Artinian.*

An easy consequence of this is that a left Noetherian (respectively left Artinian) ring which is finitely generated over its center is right Noetherian (respectively right Artinian).

Theorem 1 follows easily from Theorem 2, which gives a partial converse to the following standard fact: If $R \subset S$ are rings, and if Q is an injective R-module, then $\mathrm{Hom}_R(S, Q)$ is an injective S-module (this follows, for example, from [2: II.6, Eq. (4)]).

Theorem 2. *Let $R \subset S$ be rings such that S is finitely generated as an R-module by elements which centralize R, and let Q be a left R-module. If* $\mathrm{Hom}_R(S, Q)$ *is injective as a left S-module, then Q is injective as an R-module.*

The special case Theorem 1, a) in which both R and S are commutative was proved by P. M. Eakin in [4, Theorem 2] and by M. Nagata in [5]. Each of these proofs involves methods of ideal theory which cannot readily be extended to the non-commutative case. For various results related to Theorem 1, b), see [1].

I am very grateful to J. C. Robson for showing me that Theorem 2 can be applied to Artinian rings, and for allowing me to include his results (Theorem 1, b), and Corollary 1) in this direction.

2. Proofs and Corollaries

Proof of Theorem 2. Let E be the R-injective envelope of Q. The map $u : \mathrm{Hom}_R(S, Q) \to \mathrm{Hom}_R(S, E)$ induced by the inclusion $Q \to E$ is a monomorphism of left S-modules. $\mathrm{Hom}_R(S, Q)$ is injective by hypothesis, so u splits. We will show that u is also essential (even as an R-homomorphism), thus proving that u is an epimorphism.

The condition that S be generated by finitely many elements in the centralizer of R is easily seen to be equivalent to the condition that there exists an epimorphism of $R-R$-bimodules $R^n = \coprod_{i=1}^{n} R \twoheadrightarrow S$. Using the fact that, for any left R-module X, $\mathrm{Hom}_R(R^n, X) \cong \coprod_{i=1}^{n} X = X^n$ as left R-modules we obtain a commutative diagram of left R-modules,

$$\begin{array}{ccc} \mathrm{Hom}_R(S, Q) & \xrightarrow{\;u\;} & \mathrm{Hom}_R(S, E) \\ \downarrow & & \downarrow \\ Q^n & \longrightarrow & E^n \end{array}$$

where the vertical arrows are the monomorphisms induced by $R^n \to S$. Since $Q \to E$ is essential, so is $Q^n \to E^n$. It is easily seen that as submodules of E^n, $\mathrm{Hom}_R(S, Q) = Q^n \cap \mathrm{Hom}_R(S, E)$, and thus u is essential.

Consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(S, Q) & \xrightarrow{\;u\;} & \mathrm{Hom}_R(S, E) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(R, Q) & & \mathrm{Hom}_R(R, E) \\ \wr\| & & \wr\| \\ Q & \xrightarrow{\text{inclusion}} & E \end{array}$$

where the vertical arrows are induced by the inclusion $R \to S$. Since E is injective, the vertical arrow on the right is an epimorphism. We have already shown that u is an epimorphism, and it follows that $Q \to E$ is an epimorphism too. Thus Q is injective. $\lrcorner$

Corollary 1. *Let $R \subset S$ be as in the theorem, and suppose that S is semisimple Artinian. Then R is semisimple Artinian.*

Proof. A ring is semisimple Artinian iff each of its modules is injective. $\lrcorner$

Proof of Theorem 1, a). By a theorem of Bass [3, Prop. 4.1], it suffices to show that if $\{Q_k\}_{k \in K}$ is any set of injective left R-modules, then $\coprod_K Q_k$ is again injective. Of course, $\mathrm{Hom}_R(S, Q_k)$ is S-injective, and since S is Noetherian, $\coprod_K \mathrm{Hom}_R(S, Q_k)$ is S-injective too. As S is a finitely generated left R-module,

$$\coprod_K \mathrm{Hom}_R(S, Q_k) = \mathrm{Hom}_R(S, \coprod_K Q_k).$$

An application of Theorem 2 now finishes the proof.

Proof of Theorem 1, b). Let $N = \mathrm{Rad}(S)$, so that S/N is semisimple. By Corollary 1, $R/(R \cap N)$ is semisimple Artinian, and since N is nilpotent, so is $R \cap N$. But R is Noetherian by part a), and it follows that R is Artinian.

Corollary 2. *Let S be a ring which is finitely generated as a module over its center. If S is left Noetherian (respectively left Artinian), then S is right Noetherian (respectively right Artinian).*

Proof. By Theorem 1, Center (S) is Noetherian (respectively Artinian). The well known converse of Theorem 1 finishes the proof. ⌟

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(Received May 16, 1969)