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Hereditary Noetherian Prime Rings

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In the study of hereditary Noetherian rings, it is clear that hereditary Noetherian prime rings will play a central role (see, for example, [12]). Here we study the (two-sided) ideals of an hereditary Noetherian prime ring and, as a consequence, ascertain the structure of factor rings and torsion modules. The torsion theory represents a generalization of similar results about Dedekind prime rings ([3], Section 3).

The basic results are concerned with ideals and come in Sections 1, 2, and 4. Each ideal is a product of an invertible ideal and an ideal some power of which is idempotent; the invertible ideals generate an Abelian group; and a maximal invertible ideal is either a maximal ideal or else a finite intersection of idempotent maximal ideals of a specified form.

We will say that a ring has *enough invertible ideals* if every nonzero ideal contains an invertible ideal. All the examples of hereditary Noetherian prime rings of which we know have enough invertible ideals. They are described in Section 5.

In Section 3 we show that every finitely generated torsion module of an hereditary Noetherian prime ring with enough invertible ideals is a direct sum of cyclic modules. The proof involves showing that each factor ring is generalized uniserial. In Section 6 it is shown that a factor ring of an arbitrary hereditary Noetherian prime ring is the direct product of two rings, one generalized uniserial, the other generalised triangular.

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The results on invertible ideals and idempotent ideals in Sections 1 and 2 represent generalizations of work of Harada [5] whose results are for the case of a bounded hereditary Noetherian prime ring.

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We assume that the reader is familiar with the material of [3] Section 1, excluding the results about uniform right ideals. Unless we specifically state the contrary, conditions on rings are always meant to hold on both sides; for example, when we say that all the ideals of a ring are projective and finitely generated, we mean that they are projective and finitely generated both as right and as left ideals.

1. Over-Rings and Idempotent Ideals

Let R be an order in a quotient ring Q and let A, B be subsets of Q. We will use the notation

$$A \cdot . B = \{q \in Q \mid Aq \subseteq B\}, \qquad B . \cdot A = \{q \in Q \mid qA \subseteq B\},$$

and, for a fractional right R-ideal I, we will write $R . \cdot I = I^*$. An ideal X of R is *invertible* if $X(X \cdot . R) = (R . \cdot X) X = R$ and then we write $X \cdot . R = R . \cdot X = X^{-1}$.

It will turn out that invertible ideals and idempotent ideals play an important role in hereditary Noetherian prime rings. The main result of this section is Theorem 1.6 which shows that, for an invertible ideal X in an hereditary Noetherian prime ring R, there is a one-to-one correspondence between the idempotent ideals of R containing X and over-rings of R contained in X^{-1}. In the next section, we will use this result to examine the invertible ideals of R.

We do not need the full force of the assumption that R is an hereditary Noetherian prime ring in order to obtain Theorem 1.6. In fact, after Theorem 1.2, we will assume only that R is an order in a simple Artinian ring Q such that each ideal of R is projective.

We begin with an internal characterization of Dedekind prime rings which helps to explain the importance of idempotent ideals in an arbitrary hereditary Noetherian prime ring. But first, an easy technical lemma.

Lemma 1.1. *Let R be an order in a simple Artinian ring, and let I be a projective fractional right R-ideal. Then*

(i) *I^*I is an idempotent ideal of R*
(ii) $I = (I^* \cdot . R)$

Proof. (i) By [3] Lemma 1.2, $II^* = O_l(I)$ and so

$$(I^*I)(I^*I) = I^*O_l(I)I = I^*I.$$

(ii) Evidently $I \subseteq (I^* \cdot . R)$. But

$$I^* \cdot . R \subseteq O_l(I)(I^* \cdot . R) = II^*(I^* \cdot . R) \subseteq IR = I. \quad \blacksquare$$

THEOREM 1.2. *An hereditary Noetherian prime ring R is a Dedekind prime ring if and only if it has no proper idempotent ideals.*

Proof. $\Rightarrow$: If R is a Dedekind prime ring, then by [11], Theorem 3.2, the nonzero ideals of R form a group, and hence R can have no proper idempotent ideals.

$\Leftarrow$: Let X be any ideal of R. By [11] Theorem 2.1, it suffices to show that X is invertible. But Lemma 1.1 shows that $(R . \cdot X) X$ and $X(X \cdot . R)$ are idempotent ideals. Hence $(R . \cdot X) X = X(X \cdot . R) = R$, so X is invertible. $\blacksquare$

For the remainder of this section, R will denote an order in a simple Artinian ring Q, such that each ideal of R is projective. By [3] Lemma 1.2, this implies that each ideal is finitely generated on each side.

PROPOSITION 1.3. *Every two-sided submodule V of X^{-1} which contains R is a fractional R-ideal, and is projective and finitely generated on each side.*

Proof. By the symmetry of the situation, it suffices to prove the right-handed properties. Let $x \in X$ be a regular element, so that $xR \subseteq xV \subseteq R$. Thus xV is an essential right ideal of R, so V is a fractional right R-ideal.

Now VX is an ideal of R, and hence is projective by assumption. So, by [3] Lemma 1.2, $VX(VX)^* = O_l(VX)$. On the other hand, it suffices by [3] Lemma 1.2 to show that $VV^* = O_l(V)$. Clearly, $VV^* \subseteq O_l(V)$, and equally clearly, $O_l(V) = O_l(VX)$. Hence it suffices to show that $X(VX)^* \subseteq V^*$; that is, that $[X(VX)^*] V \subseteq R$. But $X^{-1}[X(VX)^*] VX \subseteq R \cdot R = R$, so $X(VX)^*V \subseteq XRX^{-1} = R$. $\blacksquare$

In [3] Theorem 1.3, a restricted minimum condition is obtained for an hereditary Noetherian prime ring. Under our weaker hypothesis, we still obtain a restricted minimum condition, but this time only for ideals above a fixed invertible ideal.

PROPOSITION 1.4. *If $X \subseteq R$ is an invertible ideal, then any descending chain of ideals $R \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq X$ must stabilize.*

Proof. We obtain an ascending chain $R \subseteq I_1^* \subseteq I_2^* \subseteq \cdots \subseteq X^* = X^{-1}$. Let $V = \bigcup_{k=1}^{\infty} I_k^*$. Then $R \subseteq V \subseteq X^{-1}$, so by Proposition 1.3, V is finitely generated. Thus the ascending chain stabilizes, and by Lemma 1.1 (ii), the descending chain stabilizes too. ▮

LEMMA 1.5. *Let A be an idempotent ideal of R. Then $A^*A = A$ and $A^* = O_l(A)$.*

Proof. $A^*A = A^*A^2 \subseteq RA = A \subseteq A^*A$, so $A^*A = A$. Therefore $A^* \subseteq O_l(A)$. The other containment is trivial. ▮

THEOREM 1.6. *Let R be an order in a simple Artinian ring such that each ideal of R is projective, and let X be an invertible ideal of R. Then there is a one-to-one correspondence between idempotent ideals A such that $X \subseteq A \subseteq R$ and the rings S such that $R \subseteq S \subseteq X^{-1}$, which is given by*

$$A \mapsto O_l(A) = A^*; \qquad S \mapsto S^{\cdot} . R.$$

Similarly, there is a one-to-one correspondence given by

$$A \mapsto O_r(A); \qquad S \mapsto R .^{\cdot} S.$$

Proof. If $X \subseteq A \subseteq R$, then $R \subseteq A^* = O_l(A) \subseteq X^{-1}$. Also, by Lemma 1.1(ii), $A^{*\cdot} . R = A$.

On the other hand, suppose $R \subseteq S \subseteq X^{-1}$. By Proposition 1.3, S is a finitely generated projective fractional R-ideal, so that $S(S^{\cdot} . R)$ is an idempotent ideal of R.

Clearly, $S^{\cdot} . R = S(S^{\cdot} . R)$ and $X \subseteq S^{\cdot} . R \subseteq R$. By Lemma 1.1(ii), $(S^{\cdot} . R)^* = S$. ▮

We spend the remainder of this section detailing the properties of this correspondence.

PROPOSITION 1.7. *Let A, B be idempotent ideals between X and R. Then*

(i) *$A + B$ is again idempotent, and $O_l(A + B) = O_l(A) \cap O_l(B)$.*
(ii) *If $A \subseteq B$, then $O_l(A) \supseteq O_l(B)$.*

Proof. (i) $(A + B)^2 = A + AB + BA + B \supseteq A + B$, so $A + B$ is idempotent. Clearly, $O_l(A) \cap O_l(B) \subseteq O_l(A + B)$. But it is also clear that $A^* \cap B^* \supseteq (A + B)^*$.

(ii) If $A \subseteq B$, then $A^* \supseteq B^*$. ▮

PROPOSITION 1.8. *Let A be an idempotent ideal of R such that $X \subseteq A \subseteq R$. There is a one-to-one correspondence between idempotent ideals B of R such that $B \subseteq A$, and idempotent ideals C of $S = O_l(A)$ given by*

$$B \mapsto BS \qquad C \mapsto CA.$$

Proof. Let C be an idempotent ideal of $S = A^*$. Then $CA \subseteq A^*A = A$ is an ideal of R. But $AC = AA^*C = SC = C$, so $(CA)^2 = CACA = CA$. Finally, $CAS = CAA^* = CS = C$.

On the other hand, let B be an idempotent ideal of R with $B \subseteq A$. Then

$$BS = BA^* \subseteq AA^* = S, \qquad \text{and} \qquad AB = BA = B.$$

Now

$$\begin{aligned}(BA^*)^2 &= BA^*BA^* = (BA)\, A^*(AB)\, A^* = B(AA^*A)\, BA^* \\ &= BABA^* = BA^*,\end{aligned}$$

so $BS = BA^*$ is idempotent. Finally,

$$BA^*A = (BA)\, A^*A = BA = B. \quad \blacksquare$$

PROPOSITION 1.9. *If S is a ring such that $R \subseteq S \subseteq X^{-1}$, then S is an order in Q and the ideals of S are projective. If R is an hereditary Noetherian prime ring, then so is S.*

Proof. S is an order because R is. Let I be an ideal of S. Clearly I is a fractional R-ideal, so I is projective as an R-module. Thus $I(R \,.\, ^{\cdot} I) = O_l(I)$. But $(R \,.\, ^{\cdot} I) \subseteq (S \,.\, ^{\cdot} I)$, so $I(S \,.\, ^{\cdot} I) = O_l(I)$ too.

Now suppose that R is an hereditary Noetherian prime ring. The same argument used above shows that each essential right ideal of S is projective, and therefore finitely generated. Since every right ideal of S is a direct summand of an essential right ideal, this completes the argument. $\blacksquare$

2. INVERTIBLE IDEALS

We now study the invertible ideals themselves. The main result, which is basic to the remainder of this paper, is that the invertible ideals in an hereditary Noetherian prime ring generate an Abelian group.

We will assume throughout this section that R is an order in a simple Artinian ring Q such that the ideals of R are projective. By [3] Lemma 1.2, the ideals are finitely generated on each side and so R satisfies the ascending chain condition for ideals.

PROPOSITION 2.1. *Every invertible ideal of R is a product of maximal invertible ideals (ideals maximal amongst the invertible ideals).*

Proof. If X is invertible and $P \supseteq X$ is a maximal invertible ideal then $X = PP^{-1}X$. Evidently $P^{-1}X$ is invertible and, since $R \supseteq P^{-1}X \supseteq X$, the ascending chain condition for ideals gives the desired result. $\blacksquare$

PROPOSITION 2.2. *Each maximal ideal M of R is either idempotent or invertible.*

Proof. If both $(R\,.^{\cdot}\,M)M$ and $M(M^{\cdot}\,.\,R)$ equal R, then M is invertible by definition. Otherwise one of them equals M and then M is idempotent by Lemma 1.1(i). ∎

This proposition shows that a maximal invertible ideal is either a maximal ideal or else the maximal ideals which contain it are all idempotent. We will prove shortly that each maximal invertible ideal is the intersection of the maximal ideals containing it.

LEMMA 2.3. *Let X be an invertible ideal of R and let $M \supset X$ be an idempotent maximal ideal. Then there is an idempotent maximal ideal $M' \supset X$ such that $O_r(M) = O_l(M')$.*

Proof. Let $S = O_r(M)$ and $M' = S^{\cdot}\,.\,R$. By Theorem 1.6, S is a ring between R and X^{-1} and, again by Theorem 1.6, M' is an idempotent ideal of R, $M' \supset X$ and $O_l(M') = S$. By Proposition 1.7, S is a minimal overring of R and so M' is a maximal idempotent ideal of R.

If M' is not a maximal ideal, then $M' \subset P$ where P is a maximal ideal and of course, P is invertible. But then, for each n, $P^n \supset M' \supset X$ and so, by Proposition 1.4, we must have $P^n = P^{n+1}$ for some n. Since P is invertible, this implies that $P = R$, which is a contradiction. Therefore M' is a maximal ideal. ∎

PROPOSITION 2.4. *Let X be an invertible ideal of R and let $M_1 \supset X$ be an idempotent maximal ideal. Then $X \subseteq M_1 \cap M_2 \cap \cdots \cap M_n$ where M_i is an idempotent maximal ideal and $O_r(M_1) = O_l(M_2)$, $O_r(M_2) = O_l(M_3),\ldots,$ $O_r(M_n) = O_l(M_1)$.*

Proof. Applying Lemma 2.3 to M_i yields M_{i+1}. The chain

$$M_1 \supseteq M_1 \cap M_2 \supseteq M_1 \cap M_2 \cap M_3 \supseteq \cdots \supseteq X$$

must stabilise, by Proposition 1.4. This can happen only if two M_i's coincide. Say $M_i = M_{n+i}$ is the first coincidence. If $i > 1$, then $O_r(M_{n+i-1}) = O_l(M_{n+i}) = O_l(M_i) = O_r(M_{i-1})$ and so, by Theorem 1.6, $M_{n+i-1} = M_{i-1}$. It must therefore be the case that $M_{n+1} = M_1$ is the first coincidence. ∎

A finite set of distinct idempotent maximal ideals $M_1,\ldots, M_n$ such that $O_r(M_1) = O_l(M_2),\ldots, O_r(M_n) = O_l(M_1)$ is called a *cycle*. We will also consider an invertible maximal ideal to be a trivial case of a cycle.

PROPOSITION 2.5. *Let $M_1, \ldots, M_n$ be a union of cycles of R. Then $X = \bigcap_{k=1}^{n} M_k$ is invertible.*

Proof. If M_k is idempotent, let $S_k = O_l(M_k)$ and let $M_{k'}$ be the idempotent maximal ideal such that $O_r(M_{k'}) = S_k$. By Lemma 1.5, $M_k{}^* = S_k$ and so $M_k S_k = S_k$. Also, of course, $M_{k'} S_k = M_{k'}$. Let A be the product of all the M_i other than M_k, $M_{k'}$. Then

$$R \supseteq M_{k'} S_k \supseteq X S_k \supseteq A M_{k'} M_k S_k = A M_{k'}$$

so that $S_k \subseteq X^{\cdot} . R$ and $X(X^{\cdot} . R) \not\subseteq M_k$.

If, on the other hand, M_k is invertible, let B be the product of all the M_i other than M_k. Then

$$R = M_k M_k^{-1} \supseteq X M_k^{-1} \supseteq B M_k M_k^{-1} = B,$$

so that $M_k^{-1} \subseteq X^{\cdot} . R$ and $X(X^{\cdot} . R) \not\subseteq M_k$.

Thus we see that $X(X^{\cdot} . R)$, which contains X, is not contained in any of the maximal ideals containing X. Therefore $X(X^{\cdot} . R) = R$. Similarly, $(R .^{\cdot} X) X = R$ and so X is invertible. ∎

THEOREM 2.6. *Let R be an order in a simple Artinian ring Q such that each ideal of R is projective. Then a maximal invertible ideal is the intersection of a cycle.*

Proof. This is clear from Proposition 2.4 and Proposition 2.5. ∎

Next we will show that the invertible ideals generate an Abelian group.

PROPOSITION 2.7. *Two cycles of R either coincide or are disjoint.*

Proof. Clearly we can assume the cycles are of idempotent maximal ideals. By Proposition 2.5, the intersection of the union of the two cycles is an invertible ideal. Using Theorem 1.6, the result follows easily. ∎

PROPOSITION 2.8. *Let $I = M_1 \cap \cdots \cap M_n$ where the M_i are maximal ideals of R, and let X be an invertible ideal such that $X \not\subseteq M_i$ for any i. Then $XI = X \cap I = IX$.*

Proof. Let $J = X \cap I \subseteq X$. So $J = XX^{-1}J$ and $X^{-1}J$ is an ideal of R. Now for each i, $XX^{-1}J \subseteq M_i$ and $X \not\subseteq M_i$. Thus $X^{-1}J \subseteq M_i$ for each i and so $X^{-1}J \subseteq I$. Therefore $J = X \cap I \subseteq XI$. Since $XI \subseteq X \cap I$ we have $XI = X \cap I$, and symmetry completes the proof. ∎

THEOREM 2.9. *Let R be an order in a simple Artinian ring such that each* *ideal of R is projective. Then the invertible ideals of R generate an Abelian group.*

Proof. By Proposition 2.1, each invertible ideal is a product of maximal invertible ideals. Each maximal invertible ideal is the intersection of a cycle. By Proposition 2.7, no two cycles can have an ideal in common without coinciding. Therefore, using Proposition 2.8, we see that the product of two maximal invertible ideals is commutative. ▮

COROLLARY 2.10. *Let $X = \prod_{i=1}^{n} P_i^{k_i}$ where the P_i are distinct maximal invertible ideals of R. Then $R/X \cong \prod_{i=1}^{n} R/P_i^{k_i}$ as rings.*

Proof. By Proposition 2.8, $X = \bigcap_{i=1}^{n} P_i^{k_i}$. But also, for each P_i, $(\bigcap_{i \neq j} P_i^{k_i}) + P_j^{k_j} = R$, since no maximal ideal contains both $\bigcap_{i \neq j} P_i^{k_i}$ and $P_k^{k_j}$. ▮

COROLLARY 2.11. *If R has enough invertible ideals, then the invertible fractional R-ideals form an Abelian group.*

Proof. Let X be an invertible fractional R-ideal. There is an ideal $Y \subseteq R$ such that $XY \subseteq R$ and, by our hypothesis, we may take Y to be invertible. Then XY is invertible and so X is in the Abelian group generated by the invertible ideals of R. ▮

3. FACTOR RINGS AND TORSION MODULES

We are now ready to prove that any finitely generated torsion module over an hereditary Noetherian prime ring which has enough invertible ideals is a direct sum of cyclic modules. This, together with the theory summarized in [3] Section 2 yields a survey of all finitely generated modules over such a ring. The focal point of this discussion is Theorem 3.3 which describes the structure of the factor ring of an hereditary Noetherian prime ring by an invertible ideal.

We begin by recalling a result from [3].

THEOREM 3.1. *Suppose that R is an hereditary Noetherian prime ring with enough invertible ideals. Then every finitely generated torsion R-module is a direct sum of a completely faithful module and an unfaithful module. Moreover, any completely faithful module is cyclic.*

Proof. The proofs of Lemma 3.1, Lemma 3.10, and Theorem 3.9 of [3] may be used unaltered to yield this theorem. ▮

It remains only to show that a finitely generated unfaithful module over such a ring R is a direct sum of cyclics. Being unfaithful, it has an annihilator which, by our assumptions, contains an invertible ideal X, and by [3] Theorem 1.3, R/X is Artinian. We can regard the module as an R/X-module. Thus it will be more than sufficient to show that every R/X-module is a direct sum of cyclics. Before proving this we recall a definition and a theorem due to Nakayama.

An Artinian ring S is called a *generalized uniserial* ring if and only if each indecomposable direct summand of the underlying right S-module of S has a unique composition series and the same is true of the underlying left S-module of S, ([9] p. 19). Nakayama proves the following theorem ([9], Theorem 17, [10], Theorem 3).

THEOREM 3.2. *An Artinian ring is generalized uniserial if and only if each left or right module is a direct sum of cyclic modules each of which has a unique composition series.*

Thus, in order to show that an unfaithful module over an hereditary Noetherian prime ring with enough invertible ideals is a direct sum of cyclics, we will prove the following theorem.

THEOREM 3.3. *Let R be an hereditary Noetherian prime ring and let X be an invertible ideal of R. Then R/X is generalized uniserial. If R has enough invertible ideals, then every proper factor ring of R is generalized uniserial.*

Proof. To see that the second statement follows from the first we note that, as a consequence of Theorem 3.2, any factor of a generalized uniserial ring is generalized uniserial. By Theorem 2.9, we may write $X = P_1^{k_1} \cdots P_n^{k_n}$, where each P_i is a maximal invertible ideal. The radical of R/X is easily seen to be P/X, where $P = P_1 \cdots P_n$ is an invertible ideal of R.

By [3] Theorem 1.3, R/X is Artinian, so it suffices to show that each indecomposable direct summand I/X of R/X has a unique composition series. We will show that

$$(*) \qquad I/X \supset (I/X)(P/X) \supset \cdots \supset (I/X)(P/X)^j \supset \cdots$$

is a composition series. [Note that $(I/X)(P/X)^j = (IP^j + X)/X$.]

We first remark that being a composition series, (*) must be a unique composition series. For, let

$$I/X \supset I_1/X \supset \cdots \supset I_j/X \supset \cdots$$

be any composition series. P/X is nilpotent, so it annihilates any simple R/X module, and thus $I_1/X \supseteq (IP + X)/X$. But $I/(IP + X)$ is simple, so $I_1/X = (IP + X)/X$. Similarly, $I_j/X = (IP^j + X)/X$.

It remains to show that (*) is a composition series. The simplicity of the first step follows from the fact that primitive idempotents of an Artinian ring are also primitive modulo the radical of the ring (see [6] Theorem 32, p. 72). For, since $I/(IP + X)$ is an R/P module, it is simple if it is indecomposable. Using the invertibility of P, it follows that $IP^i/(IP + X) P^i$ is simple for each i. However $(IP^i + X)/(IP^{i+1} + X) \cong IP^i/(IP^{i+1} + X) \cap IP^i$ and $(IP^{i+1} + X) \cap IP^i \supseteq IP^{i+1} + XP^i = (IP + X) P^i$. Therefore, for each i, $(IP^i + X)/(IP^{i+1} + X)$ is a homomorphic image of a simple module, and so is simple or zero, as required. ∎

COROLLARY 3.4. *Let R be an hereditary Noetherian prime ring with enough invertible ideals. Then every finitely generated torsion module is a direct sum of cyclic modules.*

We comment that the factor ring by an invertible ideal, although generalized uniserial, need not be a principal ideal ring (as it is for a Dedekind prime ring). An example of this failure is given in [3] Section 4.

Factor rings of a hereditary Noetherian prime ring by ideals which need not contain invertible ideals will be discussed in Section 6. Before this, we need to investigate the structure of an arbitrary ideal.

4. EVENTUAL IDEMPOTENTS

We now return to the study of ideals in an hereditary Noetherian prime ring R. The basic result in this section is that every ideal is the product of an invertible ideal and an ideal some power of which is idempotent. An ideal of this latter type we call *eventually idempotent*. For any ideal I we write

$$evI = \inf\{n > 0 \mid I^n = I^{n+1}\} = \inf\{n > 0 \mid I^n \text{ is idempotent}\}.$$

Clearly I is eventually idempotent if and only if $evI < \infty$. In this case evI is called the *degree of eventuality* of I.

One consequence of the basic result is that R has enough invertible ideals if and only if each idempotent maximal ideal belongs to a cycle. In particular, we show that this must be the case if the ring is bounded and has only a finite number of idempotent maximal ideals. We then show that such a ring is the intersection of Dedekind prime rings. This case includes all the examples of hereditary Noetherian prime rings in [4] and [7]–see Section 5.

Once again the results concerning the ideal structure of R do not require the full hypothesis that R be an hereditary Noetherian prime ring. To be precise, they require that R is an order in a simple Artinian ring, that the ideals of R are projective, that R satisfies the descending chain condition for ideals containing a fixed one and that prime ideals are maximal. However, for the sake of clarity we will assume throughout this section that R is an hereditary Noetherian prime ring.

First we prove an easy lemma.

LEMMA 4.1. *Let $X \subset R$ be an invertible ideal. Then*

(i) $\bigcap X^n = 0$, *and*

(ii) *X contains no idempotent ideal.*

Proof. (i) If $Y = \bigcap X^n \neq 0$, then R/Y satisfies the descending chain condition for ideals. Therefore, for some n, $X^n = X^{n+1}$ and so, since X is invertible, $X = R$, a contradiction.

(ii) If A is idempotent and $A \subseteq X$, then $A \subseteq X^n$ for all n. So $A \subseteq \bigcap X^n = 0$. ▮

One obvious consequence of this and Proposition 2.2 is that a maximal idempotent ideal is a maximal ideal.

THEOREM 4.2. *Let R be an hereditary Noetherian prime ring and I an ideal of R. Then $I = XA$ where X is an invertible ideal and A is an eventually idempotent ideal.*

Proof. We may suppose that I is not invertible. Let X be minimal among invertible ideals containing I. If $Y \neq R$ is an invertible ideal containing $X^{-1}I$ then $X \supset XY \supseteq I$ and, since XY is invertible, this contradicts the choice of X. Thus $I = XA$ where $A = X^{-1}I$ is an ideal not contained in any proper invertible ideal. Such an ideal A is eventually idempotent as will be shown in the following sequence of results.

PROPOSITION 4.3. *Let $M_1, \ldots, M_k$ be maximal ideals of R such that $A = M_1 \cap \cdots \cap M_k$ is not contained in any invertible ideal. Then A is eventually idempotent and $evA \leqslant k$.*

Proof. If $k = 1$, Proposition 2.2 suffices. Since A is, by hypothesis, not invertible, at least one of $(R \,.\!\cdot A)\, A$ and $A(A \cdot\!.\, R)$ is distinct from R, say $(R \,.\!\cdot A)\, A = A^*A \neq R$. By Lemma 1.1, A^*A is idempotent; so if $A^*A = A$ we are done. Thus we can assume that $A \subset A^*A \subset R$. Now each M_i is a maximal ideal and so, by the Chinese remainder theorem, $R/A \simeq \prod_i R/M_i$, the product of simple rings. This shows that any ideal containing A is the intersection of some subset of the M_i; in particular, we can assume that $A^*A = M_1 \cap \cdots \cap M_j = B$, say. Set $C = M_{j+1} \cap \cdots \cap M_k$. Then

$$BC \subseteq B \cap C = A = AA^*A = AB = (B \cap C)\, B \subseteq CB \subseteq B \cap C.$$

So $CB = A \supseteq BC$. By induction on k, we may assume that both B and C are eventually idempotent with $evB \leqslant j$, $evC \leqslant k - j$. The next lemma completes the proof of Proposition 4.3.

LEMMA 4.4. *Let B, C be eventually idempotent ideals of R such that $CB \supseteq BC$. Then CB is eventually idempotent and $evCB \leqslant evC + evB$.*

Proof. Let $b = evB$, $c = evC$. It is easy to see that $(CB)^{c+b} \supseteq (CB)^{c+b+1} \supseteq B^{c+b+1}C^{c+b+1} = B^bC^c$. On the other hand, it is clear that $B^bC^c \supseteq (CB)^{c+b}$ and so $(CB)^{c+b} = (CB)^{c+b+1}$. ∎

The next proposition completes the proof of Theorem 4.2.

PROPOSITION 4.5. *Let A be an ideal of R which is not contained in any invertible ideal. Then A is eventually idempotent. More precisely, there are only a finite number of idempotent maximal ideals $M_1, \ldots, M_k$ containing A and $A^k = (M_1 \cap \cdots \cap M_k)^k$ is idempotent.*

Proof. Since R has descending chain condition on ideals containing A the intersection of all maximal ideals containing A has the form $B = M_1 \cap \cdots \cap M_k$ and the M_i are idempotent by Lemma 4.1. Since all the primes of R are maximal, B/A is the nilpotent radical of the ring R/A so, for some $l > 0$, $B^l \subseteq A$. But $B^k = B^{k+1}$ by Proposition 4.3. Hence $B^k \supseteq A^k \supseteq B^{lk} = B^k$. ∎

COROLLARY 4.6. *Every idempotent ideal A of R has the form $(M_1 \cap \cdots \cap M_k)^k$ where $M_1, \ldots, M_k$ are the maximal ideals containing A.*

We are now in a position to give a condition equivalent to the ring having enough invertible ideals.

COROLLARY 4.7. *R has enough invertible ideals if and only if each idempotent maximal ideal belongs to a cycle.*

Proof. $\Rightarrow$. Let M be an idempotent maximal ideal which, by assumption, contains an invertible ideal A. By the proof of Lemma 2.4, M belongs to a cycle.

$\Leftarrow$. Let I be any ideal of R. Then $I = XA$ where X is invertible and A is eventually idempotent. Say $B = A^m$ is idempotent. Then $I \supseteq XB$ and it is clearly sufficient to show that B contains an invertible ideal. By Corollary 4.6, $B = (M_1 \cap \cdots \cap M_k)^k$ and, by assumption, each M_i belongs to a cycle whose intersection Y_i is invertible, by Proposition 2.5. Thus $B \supseteq (Y_1 \cdots Y_k)^k$ which is invertible. ∎

As another consequence of the preceding theory we have

THEOREM 4.8. *Let R be an hereditary Noetherian prime ring. The following are equivalent.*

(i) *R has a finite number of idempotent ideals.*
(ii) *R has a finite number of idempotent maximal ideals.*
(iii) *R has a minimal idempotent ideal.*

Proof. The equivalence of (i) and (ii) is evident from Corollary 4.6 and (iii) is an obvious consequence of (i). So we need only show that (iii) implies (ii). Let $A = (M_1 \cap \cdots \cap M_k)^k$ be the given minimal idempotent ideal where $M_1 ,..., M_k$ are idempotent maximal ideals, and let M be any other idempotent ideal. Let $I = M \cap M_1 \cap \cdots \cap M_k$. Then $I^k \subseteq A \cap M \subset A$, and so I is not eventually idempotent. By Theorem 4.2, I must be contained in an invertible ideal and thus in a maximal invertible ideal X. By Theorem 2.6, $X = P_1 \cap \cdots \cap P_l$ where $P_1 ,..., P_l$ forms a subset of M, $M_1 ,..., M_k$ and where, moreover, $P_1 ,..., P_l$ is a cycle.

Since A is idempotent, A is not contained in any invertible ideal (by Lemma 4.1) so $X \not\supseteq A$. Therefore one of the P_i must be M. Now consider the set of all subsets of $M_1 ,..., M_k$. For each subset, if there is an idempotent maximal ideal M which, together with the subset, forms a cycle then, by Proposition 2.7, M is unique. Since the number of subsets is finite, R has only a finite number of idempotent maximal ideals. ∎

THEOREM 4.9. *Let R be an hereditary Noetherian prime ring with a finite number of idempotent maximal ideals and with enough invertible ideals. Then R is a finite intersection of Dedekind prime rings.*

Proof. Let $A_1 ,..., A_n$ be the complete set of minimal idempotents of R. By assumption each A_i contains an invertible ideal. The product X of these invertible ideals is invertible and is contained in every idempotent ideal. Let $B = A_1 + \cdots + A_n$. By Proposition 1.7, B is idempotent and $O_l(B) = \bigcap_i O_l(A_i)$, $O_r(B) = \bigcap_i O_r(A_i)$. However, by Theorem 1.6, the $O_l(A_i)$ are precisely the maximal subrings of X^{-1}; and the same is true of the $O_r(A_i)$. Thus $O_l(B) = O_r(B)$ and so

$$B = BO_r(B) = BO_l(B) = BB^* = O_l(B)$$

which is ridiculous unless $B = R$. Hence $R = \bigcap_i O_l(A_i)$. By Proposition 1.9, $O_l(A_i)$ is an hereditary Noetherian prime ring; and by Proposition 1.8, $O_l(A_i)$ contains no idempotent ideals. So, by Theorem 1.2, $O_l(A_i)$ is a Dedekind prime ring. ∎

Next we discuss the case when R is bounded. First we recall two definitions. R is *right bounded* if every essential right ideal contains a nonzero ideal.

And R is *right primitive* if it has a maximal right ideal which contains no nonzero ideal or, equivalently, if it has a simple module with zero annihilator. As a consequence of [3] Theorem 1.3 we have

THEOREM 4.10. *Let R be an hereditary Noetherian prime ring. Then R is right primitive or right bounded and is both if and only if R is simple Artinian.*

Proof. If R is both right primitive and right bounded then R has a maximal right ideal which is not essential, so R has a minimal right ideal. As in [3] Lemma 1.1, R is then simple Artinian.

Suppose R is not right primitive, so that every simple R-module has a nonzero annihilator. If I is an essential right ideal of R then, by [3] Theorem 1.3, the module R/I has finite length. So it suffices to show that any module U of finite length has a nonzero annihilator. If V is a simple submodule of U then $\operatorname{ann} V = J \neq 0$ since R is right primitive; and by induction, we may assume that $\operatorname{ann}(U/V) = K \neq 0$. Since R is prime $0 \neq KJ \subseteq \operatorname{ann} U$. ∎

If R has enough invertibles we may go further.

COROLLARY 4.11. *If R has enough invertible ideals, then R is bounded or primitive.*

Proof. By Theorem 4.10, it will be sufficient to suppose that R is right bounded and prove that R is left bounded. Let I be an essential left ideal and let $a \in I$ be regular. So aR is an essential right ideal which, by hypothesis, contains an invertible ideal X. Then $Ra^{-1} \subseteq X^{-1}$ and multiplication on the right by a and on the left by X yields $X \subseteq Ra \subseteq I$. Thus R is left bounded. ∎

The next result should be compared with Theorem 4.9.

THEOREM 4.12. *Let R be a bounded hereditary Noetherian prime ring with a finite number of idempotent ideals. Then R has enough invertible ideals and is the intersection of a finite number of bounded Dedekind prime rings.*

Proof. Since R is bounded, it follows easily (see [6], pp. 120–121) that orders containing R and equivalent to R are fractional R-ideals. Thus, by an argument similar to the proof of Theorem 1.6, we can obtain the same one-to-one correspondences as those described in Theorem 1.6, but this time between all the idempotent ideals of R and all the equivalent orders containing R. Therefore, given any idempotent maximal ideal M_1, $M_2 = O_r(M_1)^{\cdot}$. R is also an idempotent maximal ideal and $O_r(M_1) = O_l(M_2)$. Since there are only a finite number of idempotent maximal ideals, this process will yield a cycle with M_1 as a member. Hence, by Corollary 4.7, R has enough invertibles.

Then Theorem 4.9 shows that R is a finite intersection of Dedekind prime rings and it is straightforward to show that they are bounded. ∎

We end this section with a result basically due to Michler [8].

THEOREM 4.13. *Let R be an hereditary Noetherian prime ring with a nonzero Jacobson radical J. Then R is bounded, R has a finite number of maximal ideals and J is invertible.*

Proof. Evidently, by Theorem 4.10, R is bounded. We will show that every maximal right ideal of R contains a maximal ideal. Then J is the intersection of the maximal ideals of R, and so they must be finite in number. It then follows by Theorem 4.12 that R has enough invertible ideals, so each maximal ideal of R belongs to a cycle and, by Proposition 2.5, J is invertible.

So it remains only to show that a maximal right ideal I contains a maximal ideal. Since R is bounded, I contains a nonzero ideal. But it is clear from the structure of ideals of R that each nonzero ideal contains a product of maximal ideals. So $I \supseteq M_1 \cdots M_n$ where the M_i are maximal ideals. If $M_1 \not\subseteq I$, then $I + M_1 = R$ and so $I \supseteq (I + M_1) M_2 \cdots M_n = M_2 \cdots M_n$. By iteration we see that I contains a maximal ideal. ∎

5. EXAMPLES

The examples of hereditary Noetherian prime rings discussed in [4] and [7] may be summarized as follows. Let D be a noncommutative Dedekind domain with a unique maximal right and left ideal M. Then any "tiled" matrix ring of the form

$$R = \begin{pmatrix} \boxed{D} & & & & D \\ & \boxed{D} & & & \\ & & \boxed{D} & & \\ & & & \ddots & \\ M & & & & \boxed{D} \end{pmatrix}$$

(squares of D's along the diagonal, with D's above and M's below) is an hereditary Noetherian prime ring. In fact, as Michler proves in [7], such a ring R is characterized as an hereditary Noetherian prime ring with a nonzero Jacobson radical J such that idempotent elements can be lifted, modulo J. By Theorem 4.13, R is bounded and has only a finite number of maximal ideals. Hence, by Theorem 4.8, R has only a finite number of idempotent ideals and, by Theorem 4.12, has enough invertible ideals.

We note next that any order R in a central simple algebra over a commutative Dedekind domain is bounded and is contained in an equivalent maximal order ([6], pp. 125–126). If R is an hereditary Noetherian prime ring, then the one-to-one correspondence discussed in the proof of Theorem 4.12 shows that, corresponding to the maximal order there is a minimal idempotent ideal in R. Therefore, by Theorem 4.8, R has a finite number of idempotent ideals and, by Theorem 4.12, R has enough invertible ideals.

We have no example of an hereditary Noetherian prime ring with an infinite number of idempotent ideals, nor of one which has not enough invertible ideals. (From the results of Section 4, it seems likely that an hereditary Noetherian prime ring with a finite number of idempotent ideals must have enough invertible ideals.) We do, however, have one example which is of interest in view of Theorem 4.10 and Corollary 4.11. It is an hereditary Noetherian prime ring R which is primitive (but is not a Dedekind prime ring).

We start by recalling Example (ii)(a) of [3], Section 4. This describes a noncommutative Dedekind domain D which is a primitive principal ideal domain with a unique maximal ideal $xD = Dx$ such that D/xD is a field. We will show that the ring

$$R = \begin{pmatrix} D & D \\ xD & D \end{pmatrix}$$

is as claimed.

First we note that R is an order in the simple Artinian ring F_2, where F is the quotient division ring of D. Thus we may use the theory of uniform right ideals outlined in [3] Section 1. It is easy to check that $U = \left(\begin{smallmatrix} D & D \\ 0 & 0 \end{smallmatrix}\right)$ is a uniform right ideal. Therefore U contains a copy of every uniform right ideal.

Let $V \subseteq U$ be a right ideal. Then $V = \left(\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right)$ where $A = aD$, $B = bD$ are right ideals of D. It is easily verified that $Bx \subseteq A \subseteq B$, i.e., $bxD \subseteq aD \subseteq bD$. But $bD/bxD \simeq D/xD$ which is a simple module, and so $aD = bD$ or $aD = bxD$. Hence

$$V = \begin{pmatrix} bD & bD \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} R \cong \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}$$

or

$$V = \begin{pmatrix} bxD & bD \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R \cong \begin{pmatrix} 0 & 0 \\ xD & D \end{pmatrix},$$

the isomorphisms being the obvious ones. Since

$$\begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ xD & D \end{pmatrix} \quad R,$$

this shows that the uniform right ideals are all principal and projective.

Thus, to show that R is hereditary and Noetherian it will be sufficient to prove that every right ideal of R is a direct sum of uniform right ideals. Let I be an arbitrary right ideal of R and let U_1 be its projection onto U, $U_1 = \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) R$ or $U_1 = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) R$. Then it follows that I contains a matrix of the form $\left(\begin{smallmatrix} b & 0 \\ c & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix}\right)$ and we write $I_1 = \left(\begin{smallmatrix} b & 0 \\ c & 0 \end{smallmatrix}\right) R$ or $I_1 = \left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix}\right)$. Now I_1 is uniform or zero and, if $I_1 \neq I$, the set of elements of I having the form $\left(\begin{smallmatrix} 0 & 0 \\ h & k \end{smallmatrix}\right)$ is a uniform right ideal I_2 such that $I_1 \oplus I_2 = I$.

Thus R is an hereditary Noetherian prime ring. (In fact, we have shown that every right ideal of R has a generating set of two elements.) If $M \neq xD$ is a maximal right ideal of D, it can be seen that $\left(\begin{smallmatrix} M & M \\ xD & D \end{smallmatrix}\right)$ is a maximal right ideal of R which contains no ideal of R. Thus R is primitive. Also R has precisely two maximal ideals $\left(\begin{smallmatrix} xD & D \\ xD & D \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} D & D \\ xD & xD \end{smallmatrix}\right)$. They are idempotent and form a cycle. By Theorem 1.2, R is not a Dedekind prime ring.

6. Arbitrary Factor Rings

We are now in a position to investigate the factor rings of an arbitrary hereditary Noetherian prime ring. We will show that such a ring is a ring direct sum of a generalized uniserial ring and a ring each of whose factor rings has finite global dimension.

In [1], Chase defines a *generalized triangular matrix ring* to be a semiprimary ring with radical N and a complete set of primitive idempotents $e_1, \dots, e_n$ such that $e_i N e_j = 0$ for $i \geqslant j$. He proves ([1], Theorem 4.1)

THEOREM 6.1. *A semiprimary ring S is a generalized triangular matrix ring if and only if* gl. dim $S/A < \infty$ *for each ideal A of S.*

Thus, we will prove that any factor ring of an hereditary Noetherian prime ring is a ring direct sum of a generalized uniserial ring and a generalized triangular matrix ring.

LEMMA 6.2. *Let R be an hereditary Noetherian prime ring, I an ideal of R. Then there is an idempotent ideal A and a invertible ideal X such that $A \cap X \subseteq I$* and $A + X = R$.

Proof. By Theorem 4.2 we may write $I = YB$, where Y is invertible and B is eventually idempotent, say $evB = b$. By Theorem 2.6, Y is a product of maximal invertible ideals, $Y = P_1 \cdot P_2 \cdots P_m$, each of which must be the intersection of a cycle. By Proposition 4.5, $C = B^b$ is a power of an intersection of maximal idempotent ideals, $C = (M_1 \cap \cdots \cap M_n)^n$, say. One of these may contain one of the factors of Y, say $M_1 \supset P_{k_1}$. Then $C \supseteq (P_{k_1} \cap M_2 \cap \cdots \cap M_n)^n$. Replacing each such M_i by a P_{k_i}, we get

$$B \supseteq C \supseteq (P_{k_1} \cap P_{k_2} \cap \cdots \cap P_{k_r} \cap M_{r+1} \cap \cdots \cap M_n)^n$$

$$= (P_{k_1} \cap P_{k_2} \cap \cdots \cap P_{k_r})^n (M_{r+1} \cap \cdots \cap M_n)^n,$$

where the last equality follows by Proposition 2.8. By Proposition 2.5, $P_{k_1} \cap \cdots \cap P_{k_r}$ is invertible. Set $X = Y(P_{k_1} \cap \cdots \cap P_{k_n})^n$ and $A = (M_{r+1} \cap \cdots \cap M_n)^n$. By Proposition 4.5, A is idempotent, and by Proposition 2.8, $XA = X \cap A$. Hence we have $I = YB \supset YC \supseteq XA = X \cap A$, as desired, and $A + X = R$ because no maximal ideal contains both A and X. ∎

THEOREM 6.3. *Let I be an ideal of the hereditary Noetherian prime ring R. Then R/I is a ring direct sum of a generalized uniserial ring and a generalized triangular matrix ring.*

Proof. By Lemma 6.2, there is an invertible ideal X and an idempotent ideal A such that $X \cap A \subseteq I$, $X + A = R$. Thus R/I is a homomorphic image of the ring $R/X \oplus R/A$, so $R/I = R/X' \oplus R/A'$, where X' and A' are ideals with $X' \supseteq X$ and $A' \supseteq A$. Now homomorphic images of generalized uniserial rings are generalized uniserial, so using Theorem 3.3, we see that R/X' is generalized uniserial. On the other hand, all the ideals containing A are eventually idempotent by Proposition 4.5. By [2] Theorem 5, factor rings of an hereditary ring by an eventually idempotent ideal have finite global dimension and, by [3] Theorem 1.3, R/A' is Artinian and therefore semiprimary. Hence by Theorem 6.1, R/A' is a generalized triangular matrix ring. ∎

It will be clear to the reader from the comments in Section 5 that we know of no example where the factor ring is not generalized uniserial.

Note added in proof. (i) In [13] Corollary 3.2 it is shown that every proper factor ring of an hereditary Noetherian prime ring is generalized uniserial. (ii) Examples are now known [14] of hereditary Noetherian prime rings which have a finite number of idempotent ideals and which do not have enough invertible ideals.

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